Lecture 2 (1/21). Please enable video if you can.

Last time: Cts time markets

→ Black Scholes formula → Replication

→ Contract R part w/ same tricks from DTF

[formula used CDF of Normal!]

Theorem 4.7 (Central limit theorem). $\boxed{S_n/\sqrt{n} \to \mathcal{N}(0,1)}$. *That is, for any bounded continuous function f,*

$$\boldsymbol{E} f\Big(\frac{S_n}{\sqrt{n}}\Big) = \boldsymbol{E} f(\mathcal{N}(0,1)) .$$

$$S_n = \sum_{1}^{n} X_k \qquad X_k \to \text{iid}, \quad EX_k = 0 . \ EX_k^2 = 1$$

Q: In what sense does $\dfrac{S_n}{\sqrt{n}} \longrightarrow \mathcal{N}(0,1)$?

(those conv)
a.s. conv

Opt ①: $\forall \omega \in \Omega$, want $\lim\limits_{n\to\infty} \dfrac{S_n(\omega)}{\sqrt{n}} \longrightarrow \mathcal{N}(0,1)(\omega)$

Mean sq conv.

Opt ②: Want $E\left|\dfrac{S_n}{\sqrt{n}} - \mathcal{N}(0,1)\right|^2 \xrightarrow{n\to\infty} 0$

Defn ③ Want the dist of $\underbrace{\frac{S_n}{\sqrt{n}}}_{\text{may be discrete RV's}}$ to conv $\longrightarrow$ $\underbrace{\text{dist}\left(N(0,1)\right)}_{\text{cts.}}$

ⓐ Take a "test function", $\underline{\underline{f}}$

$$E\, f\left(\frac{S_n}{\sqrt{n}}\right) \xrightarrow{\text{Want}} E\, f\left(\underline{N(0,1)}\right)$$

for EVERY test funtion f.

(Analogy: Say f, g are 2 fnc (cts)

$$f : [0,1] \to \mathbb{R}$$
$$g : [0,1] \to \mathbb{R}$$

Q: Does $\int_0^1 f = \int_0^1 g \Rightarrow f = g$

(NO)

Q: Does $\int_0^1 f \cdot h = \int_0^1 g \cdot h$ $\forall$ test fn $h \Rightarrow f = g$

Yes

(Reason: $\int_0^1 f\,h = \int_0^1 g\,h \quad \forall h$

$$\Rightarrow \int_0^1 (f-g)\,h = 0 \quad \forall h$$

Choose $h = \underline{f-g} \Rightarrow \underbrace{f=g}$!)

Let X be a random variable.

Definition 4.8. The *characteristic function* of X is defined by $\varphi_X(\lambda) = \boldsymbol{E} e^{i\lambda X}$.

Definition 4.9. The *moment generating function* (MGF) of X is defined by $M_X(\lambda) = \boldsymbol{E} e^{\lambda X}$.

Example 4.10. If $X \sim N(0,1)$ then $\varphi_X(\lambda) = e^{-\lambda^2/2}$, and $\boxed{M_X(\lambda) = e^{\lambda^2/2}}$.

$(i = \sqrt{-1}$

$e^{i\theta} = \cos\theta + i\sin\theta)$

Note : X is a RV.

MGF of X (Notation M_X) is a fn.

domain ~~$\mathbb{R}$~~ target $\mathbb{R}$.

subset of $\mathbb{R}$

Domain of char fn $\varphi_X = \mathbb{R}$. ($\because$ $|e^{i\lambda X}| = 1$)

MGF of Normal.

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$X \sim N(0,1).$$

$$\varphi_X(\lambda) = \mathbb{E}\, e^{\lambda X} = \int_{\mathbb{R}} e^{\lambda x} \phi_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{\lambda x} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}.$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2\lambda x + \lambda^2) + \lambda^2/2} \frac{dx}{\sqrt{2\pi}}$$

$$= e^{\lambda^2/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-\lambda)^2} \frac{dx}{\sqrt{2\pi}}$$

$y = x - \lambda$

$$= e^{\lambda^2/2} \underbrace{\int_{-\infty}^{\infty} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}}_{1}.$$

$$= e^{\lambda^2/2}.$$

QED

(You check $\varphi_X(\lambda) = e^{-\lambda^2/2}$).

Theorem 4.11. $\boldsymbol{E}X^n = (-i)^n \varphi_X^{(n)}(0) = M_X^{(n)}(0)$. *In particular,* $\boldsymbol{E}X = -i\varphi'_X(0) = M'_X(0)$, *and* $\boldsymbol{E}X^2 = -\varphi''_X(0) = M''_X(0)$.

Remark 4.12. Here $f^{(n)}(0)$ denotes the n^{th} derivative of f at 0.

Reason : $\varphi_X(\lambda) = E\, e^{i\lambda X}$ (def)

$$\frac{d}{d\lambda}\varphi_X(\lambda) = \frac{d}{d\lambda}\left(E\, e^{i\lambda X}\right)$$

$$= E\left((iX)\cdot e^{i\lambda X}\right)$$

$\Rightarrow \varphi_X'(0) = E\left((iX)\cdot 1\right) = i\,EX$. Repeat for higher mnts.

Let X, Y be two random variables.

Theorem 4.13. *The following are equivalent.*

(1) *X and Y have the same distribution (PDF)*
(2) *X and Y have the same CDF.*
(3) *X and Y have the same characteristic function.*
(4) *X and Y have the same moment generating function.*

same notion of conv for CLT.

Theorem 4.14. *A sequence of random variables $(X_n) \to X$ (in distribution) if and only if $\boxed{\varphi_{X_n} \to \varphi_X}$ pointwise.*

Theorem 4.15. *A sequence of random variables $(X_n) \to X$ (in distribution) if and only if $M_{X_n} \to M_X$ pointwise.*

Remark 4.16. The proofs of Theorem 4.13–4.15 are beyond the scope of this course; we will use them without proof.