Lecture 1 (1/19). Please enable video if you can.

2. Syllabus Overview

- Class website and full syllabus: https://www.math.cmu.edu/~gautam/sj/teaching/2021-22/420-cts-time-fin
- TA's: Jonghwa Park <jonghwap@andrew.cmu.edu>.
- Homework Due: 2:29PM, Wednesdays.
- Midterms: Wed 2/23, ~~Mon 4/4~~ ~~(closed~~ book in class). Grade scale.
- **Homework:**
  - ▷ Good quality scans please! Use a scanning app, and not simply take photos. (I use Adobe Scan.)
  - ▷ 20% penalty if turned in within an hour of the deadline. 100% penalty after that.
  - ▷ Two ~~One~~ homework assignments can be turned in 24h late without penalty.
  - ▷ Bottom two homework score is dropped from your grade (personal emergencies, interviews, other deadlines, etc.).
  - ▷ Collaboration is encouraged. Homework is not a test – ensure you learn from doing the homework.
  - ▷ You must write solutions independently, and can only turn in solutions you fully understand.
- **Academic Integrity**
  - ▷ Zero tolerance for violations (automatic **R**).
  - ▷ Violations include:
    - – Not writing up solutions independently and/or plagiarizing solutions
    - – Turning in solutions you do not understand.
    - – Seeking, receiving or providing assistance during an exam.
  - ▷ All violations will be reported to the university, and they may impose additional penalties.
- **Grading:** ~~10% homework, 30% midterm, 60% final.~~ — HW 30%, Midterm 20% (each) Final 30%

Course Outline.

- Develop tools to price securities in continuous time.
  - ▷ Brownian motion (not as easy as coin tosses)
  - ▷ Conditional expectation: No explicit formula!
  - ▷ Itô formula: main tool used for computation. Develop some intuition.
  - ▷ Measurablity / risk neutral measures: much more abstract. Complete description is technical. But we need a working knowledge.
  - ▷ Derive and understand the Black-Scholes formula.
  - ▷ Fundamental theorems of asset pricing
  - ▷ Asian options, Barrier options, etc.

3. Introduction.

(1) Binomial model: Trade at discrete time intervals (370).
(2) Suppose now we can trade *continuously in time.*
(3) Consider a market with a bank and a stock, whose spot price at time t is denoted by S_t.
(4) The *continuously compounded interest rate* is r (i.e. money in the bank grows like $\partial_t C(t) = rC(t)$.
(5) Assume liquidity, neglect transaction costs (frictionless), and the borrowing/lending rates are the same.
(6) In the *Black-Scholes* setting, we model the stock prices by a *Geometric Brownian motion* with parameters α (the mean return rate) and σ (the volatility).
(7) *(Black-Scholes Formula)* The price at time t of a European call with maturity T and strike K is given by

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)),$$

$\tau = T - t$

$$\text{where} \quad d_\pm = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right), \qquad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy.$$

(no α here)

(8) Can be obtained as the limit of the Binomial model as $N \to \infty$ by choosing:

$$r_{\text{binom}} = \frac{r}{N}, \qquad u = u_N = 1 + \frac{r}{N} + \frac{\sigma}{\sqrt{N}} \qquad d = d_N = 1 + \frac{r}{N} - \frac{\sigma}{\sqrt{N}}$$

Remark 3.1. There's no explicit formula for the option price for fixed N in the Binomial model. But there's a "nice" explicit formula when $N \to \infty$.

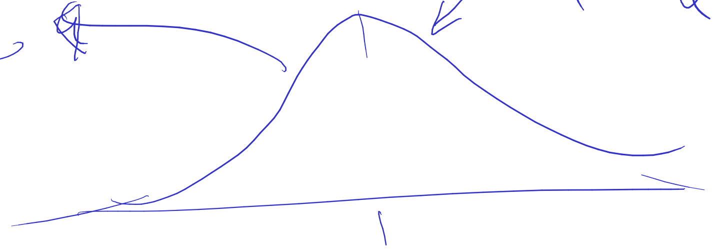
CDF of a std normal

4. Central limit theorem (review).

Definition 4.1. We say X is a *normally distributed* random variable with mean μ and variance σ^2 if the PDF of X is

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

bell curve

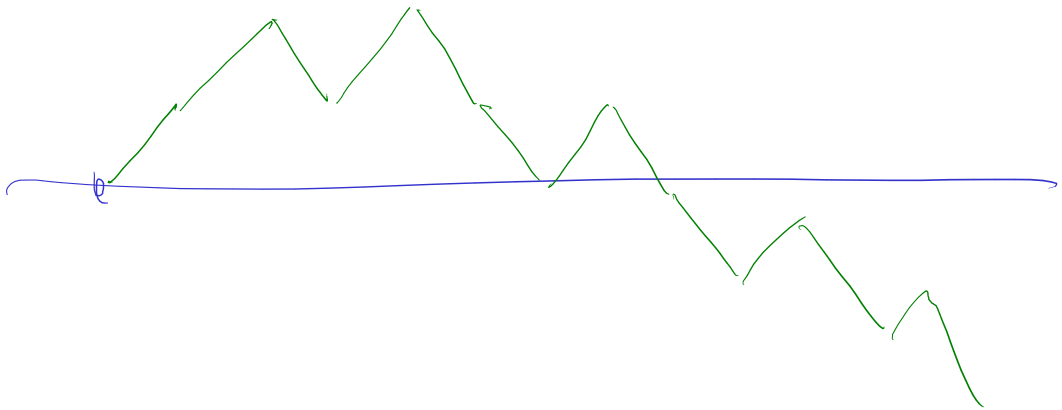


Remark 4.2. Notation: $X \sim \mathcal{N}(\mu, \sigma^2)$.

Remark 4.3. Normally distributed random variables are also called *Gaussian*.

μ — mean $\to$ $\mu = \mathbb{E} X$

$\sigma \to$ var $\qquad \sigma^2 = \mathbb{E}(X-\mu)^2$



Let $X_1, \ldots, X_n$ be a sequence of i.i.d. random variables, with $\boldsymbol{E} X_n = 0$ and $\operatorname{Var} X_n = 1$. Let $S_0 = 0$, $S_n = \sum_{k=1}^n X_k$.

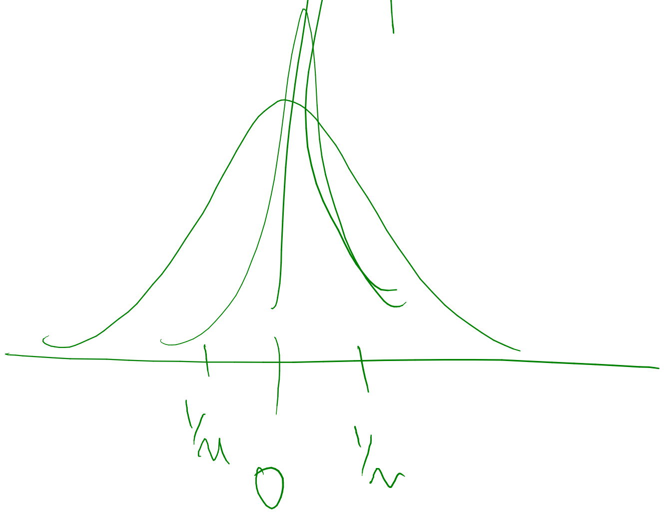
Question 4.4. *How does S_n behave as $n \to \infty$.*

Theorem 4.5 (Law of large numbers)**.** $\boxed{S_n/n \to 0 \text{ as } n \to \infty}$.

Remark 4.6. Easy check: Compute $\operatorname{Var}(S_n/n)$.

Compute :
$$\frac{E\, S_n}{n} = \frac{1}{n} E\left(\sum_1^n X_k\right) = 0$$

② Compute
$$\operatorname{Var}\left(\frac{S_n}{n}\right) = \frac{1}{n^2}\operatorname{Var}(S_n)$$
$$= \frac{1}{n^2}\operatorname{Var}\left(\sum_1^n X_k\right)$$
$$= \frac{1}{n^2}\sum_1^n \operatorname{Var}(X_k) = \frac{n}{n^2} = \boxed{\frac{1}{n}}$$



Theorem 4.7 (Central limit theorem). $\boxed{S_n/\sqrt{n}} \to \underset{\text{(std normal)}}{\mathcal{N}(0,1)}$. *That is, for any bounded continuous function* f,

$$\lim_{n\to\infty} \boxed{\boldsymbol{E} f\left(\frac{S_n}{\sqrt{n}}\right)} = \boxed{\boldsymbol{E} f\big(\mathcal{N}(0,1)\big)}.$$

(Note $\quad \mathrm{Var}\left(\frac{S_n}{\sqrt{n}}\right) = \frac{1}{n}\,\mathrm{Var}(S_n) = \frac{n}{n} = 1$

Note $\quad E f(\mathcal{N}(0,1)) = \int_{x\in\mathbb{R}} f(x) \underbrace{p(x)}_{\text{pdf}}\, dx$

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$