Continuous Time Finance: Final.

2022-05-09

- *This is a closed book test. You may not use phones, calculators, or other electronic devices.*
- *You may not give or receive assistance.*
- *You have 3 hours. The exam has a total of 8 questions and 80 points.*
- *The questions are roughly ordered by difficulty. Good luck.*

In this exam W always denotes a standard Brownian motion, and the filtration $\{\mathcal{F}_t \mid t \geqslant 0\}$ is the Brownian filtration. Here are a few formulae that you can use:

- Solution formula to the Black Scholes PDE:

$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\Big(x \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\, y\Big)\Big) \frac{e^{-y^2/2} dy}{\sqrt{2\pi}}\,, \qquad \tau = T - t\,.$$

- Black Scholes Formula for European calls, and the Greeks

$$c(t,x) = xN(d_+) - Ke^{-r\tau}N(d_-) \qquad d_\pm \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \qquad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\, dy\,,$$

$$\partial_x c = N(d_+)\,, \qquad \partial_x^2 c = \frac{1}{x\sigma\sqrt{2\pi\tau}} \exp\Big(\frac{-d_+^2}{2}\Big)\,, \qquad \partial_t c = -rKe^{-r\tau}N(d_-) - \frac{\sigma x}{\sqrt{8\pi\tau}} \exp\Big(\frac{-d_+^2}{2}\Big)\,.$$

[10] 1. Let $X \sim \mathcal{N}(0,1)$ be a standard normal. Compute $\displaystyle\int_0^\infty t\boldsymbol{P}(X^2 \geqslant t)\, dt$. Express your answer without involving expectations or integrals.

[10] 2. Let X, Z be two independent random variables, and set $Y = XZ$. Suppose $X \sim \mathcal{N}(0,1)$, and $\boldsymbol{P}(Z = 1) = 1/3$, $\boldsymbol{P}(Z = 2) = 2/3$. Find the characteristic function of Y. That is, given $\lambda \in \mathbb{R}$ compute $\boldsymbol{E}e^{i\lambda Y}$, where $i = \sqrt{-1}$. Express your answer in terms of λ without involving expectations or integrals.

[10] 3. Let B be a Brownian motion that is independent of W. Compute $\boldsymbol{E}_s\Big[\Big(\displaystyle\int_0^t B_r\, dW_r\Big)^2\Big]$. Express your answer without using expectations or conditional expectations.

[10] 4. Consider a market with a bank and a dividend paying stock. The interest rate is r, and the stock price is modelled by $dS_t = (\alpha - a)S_t\, dt + \sigma S_t\, dW_t$. Here $\alpha, \sigma > 0$ are constants, and $a > 0$ is the rate at which the stock pays dividends. Consider a forward contract with delivery price K and maturity time T. Find the arbitrage free price of this contract at time $t \leqslant T$. Also find the composition of the replicating portfolio at time t (i.e. find the amount the replicating portfolio holds in cash, and the amount it invests in the stock at time t).

[10] 5. Let $W = (W^1, W^2)$ be a standard 2-dimensional Brownian motion. Let $B^1 = 2W^1 + 3W^2$ and $B^2 = W^1 + \alpha W^2$. Find $\alpha \in \mathbb{R}$ such that B^1 and B^2 independent. You should prove B^1 and B^2 are independent for the α you find.

[10] 6. Consider a market with a bank and a stock. The interest rate is r and the stock price is modelled by a geometric Brownian motion with mean return rate α and volatility σ. (The stock does not pay any dividends.) A digital share option with strike $K \geqslant 0$ and maturity time T gives the holder one share at maturity, *provided* the stock price is larger than the strike price. The option pays nothing otherwise. Find the arbitrage free price of this option at time $t \leqslant T$. Express your answer in the form $V_t = f(t, S_t)$, and find an explicit formula for the function f without using expectations or integrals. (Your formula may involve the CDF of the standard normal.)

[10] 7. Consider a market with a bank and two stocks, whose prices are denoted by S^1 and S^2 respectively. The bank has interest rate r, and the stock prices satisfy

$$dS_t^1 = \alpha_1 S_t^1\, dt + \sigma_1 S_t^1\, dW_t \qquad \text{and} \qquad dS_t^2 = \alpha_2 S_t^2\, dt + \sigma_2 S_t^2\, dW_t\,,$$

respectively. (The stocks do not pay dividends.) Suppose $\sigma_2(\alpha_1 - r) > \sigma_1(\alpha_2 - r)$. Explicitly find an arbitrage opportunity in this market.

[This was something we did in class. Please provide a complete solution here that does not rely on the result from class.]

$\boxed{10}$ 8. Let $W = (W^1, W^2)$ be a standard 2-dimensional Brownian motion. Let $X_t = t + W_t^1 + W_t^2$, and $Y_t = e^{-t} + W_t^2$. Is there an equivalent measure $\tilde{\boldsymbol{P}}$ under which the processes X and Y are independent? If yes, find it. If no, prove it. (Correctly guessing yes/no is worth 2 points. Finding the equivalent measure or proving it doesn't exist is worth 8 points.)