LECTURE NOTES ON CONTINUOUS TIME FINANCE
FALL 2022

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Lecture 1 (1/19). Please enable video if you can.

2. Syllabus Overview

- Class website and full syllabus: https://www.math.cmu.edu/~gautam/sj/teaching/2021-22/420-cts-time-fin
- TA's: Jonghwa Park <jonghwap@andrew.cmu.edu>.
- Homework Due: 2:29PM, Wednesdays. Gradscope.
- Midterms: Wed 2/23, ~~Mon 4/4 (closed~~ book in class).
- **Homework:**
  - ▷ Good quality scans please! Use a scanning app, and not simply take photos. (I use Adobe Scan.)
  - ▷ 20% penalty if turned in within an hour of the deadline. 100% penalty after that.
  - ▷ Two ~~One~~ homework assignments can be turned in 24h late without penalty.
  - ▷ Bottom two homework score is dropped from your grade (personal emergencies, interviews, other deadlines, etc.).
  - ▷ Collaboration is encouraged. Homework is not a test – ensure you learn from doing the homework.
  - ▷ You must write solutions independently, and can only turn in solutions you fully understand.
- **Academic Integrity**
  - ▷ Zero tolerance for violations (automatic **R**).
  - ▷ Violations include:
    - – Not writing up solutions independently and/or plagiarizing solutions
    - – Turning in solutions you do not understand.
    - – Seeking, receiving or providing assistance during an exam.
  - ▷ All violations will be reported to the university, and they may impose additional penalties.
- **Grading:** ~~10% homework, 30% midterm, 60% final.~~ — HW 30%, Midterm 20% (each) Final 30%

Course Outline.

- Develop tools to price securities in continuous time.
  - ▷ Brownian motion (not as easy as coin tosses)
  - ▷ Conditional expectation: No explicit formula!
  - ▷ Itô formula: main tool used for computation. Develop some intuition.
  - ▷ Measurablity / risk neutral measures: much more abstract. Complete description is technical. But we need a working knowledge.
  - ▷ Derive and understand the Black-Scholes formula.
  - ▷ Fundamental theorems of asset pricing
  - ▷ Asian options, Barrier options, etc.

3. Introduction.

(1) Binomial model: Trade at discrete time intervals (370).

(2) Suppose now we can trade *continuously in time.*

(3) Consider a market with a bank and a stock, whose spot price at time t is denoted by S_t.

(4) The *continuously compounded interest rate* is r (i.e. money in the bank grows like $\partial_t C(t) = rC(t)$.

(5) Assume liquidity, neglect transaction costs (frictionless), and the borrowing/lending rates are the same.

(6) In the *Black-Scholes* setting, we model the stock prices by a *Geometric Brownian motion* with parameters α (the mean return rate) and σ (the volatility).

(7) *(Black-Scholes Formula)* The price at time t of a European call with maturity T and strike K is given by

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)),$$

$\tau = T - t$

$$\text{where} \quad d_\pm = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right), \qquad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy\,.$$

(no α here)

(8) Can be obtained as the limit of the Binomial model as $N \to \infty$ by choosing:

$$r_{\text{binom}} = \frac{r}{N}, \qquad u = u_N = 1 + \frac{r}{N} + \frac{\sigma}{\sqrt{N}} \qquad d = d_N = 1 + \frac{r}{N} - \frac{\sigma}{\sqrt{N}}$$

Remark 3.1. There's no explicit formula for the option price for fixed N in the Binomial model. But there's a "nice" explicit formula when $N \to \infty$.

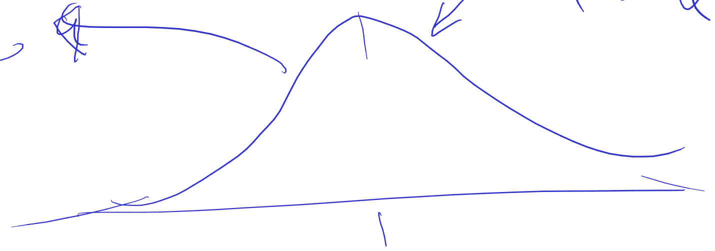
CDF of a std normal

4. Central limit theorem (review).

Definition 4.1. We say X is a *normally distributed* random variable with mean μ and variance σ^2 if the PDF of X is

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

bell curve

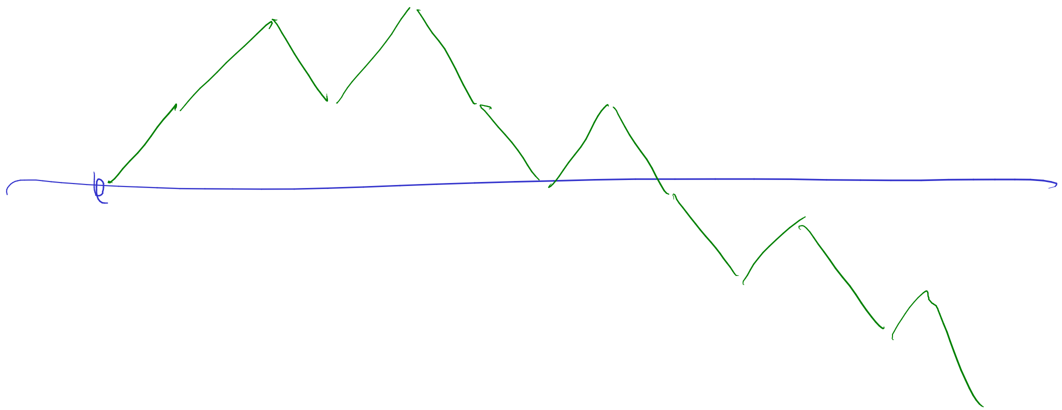


Remark 4.2. Notation: $X \sim \mathcal{N}(\mu, \sigma^2)$.

Remark 4.3. Normally distributed random variables are also called *Gaussian*.

μ — mean $\rightarrow$ $\mu = EX$

$\sigma \rightarrow$ var $\sigma^2 = E(X-\mu)^2$



Let $X_1, \ldots, X_n$ be a sequence of i.i.d. random variables, with $\boldsymbol{E} X_n = 0$ and $\operatorname{Var} X_n = 1$. Let $S_0 = 0$, $S_n = \sum_{k=1}^{n} X_k$.

Question 4.4. *How does S_n behave as $n \to \infty$.*

Theorem 4.5 (Law of large numbers). $\boxed{S_n/n \to 0 \text{ as } n \to \infty}$.

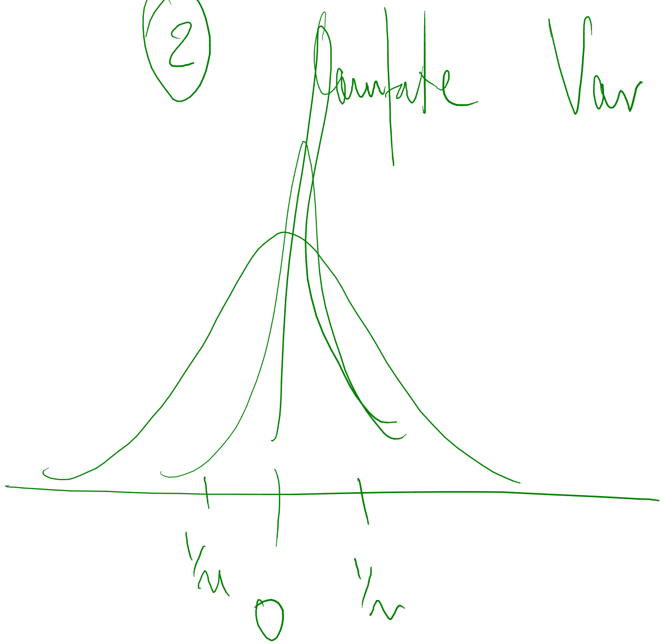
Remark 4.6. Easy check: Compute $\operatorname{Var}(S_n/n)$.

Compute : $\quad \dfrac{E\, S_n}{n} = \dfrac{1}{n} E\left(\sum_1^n X_k\right) = 0$

(2) Compute $\operatorname{Var}\left(\dfrac{S_n}{n}\right) = \dfrac{1}{n^2}\operatorname{Var}(S_n)$

$$= \frac{1}{n^2}\operatorname{Var}\left(\sum_1^n X_k\right)$$

$$= \frac{1}{n^2}\sum_1^n \operatorname{Var}(X_k) = \frac{n}{n^2} = \boxed{\frac{1}{n}}$$



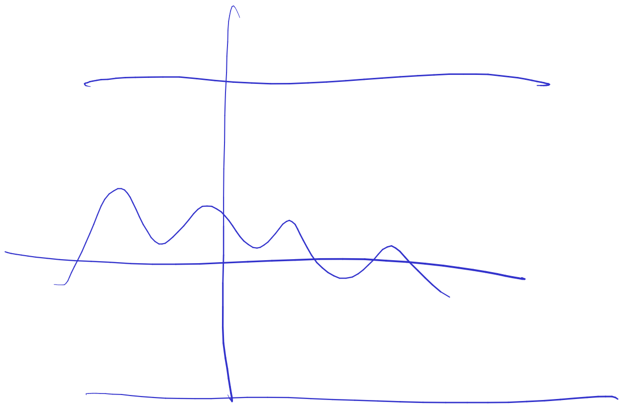
Theorem 4.7 (Central limit theorem). $\boxed{S_n/\sqrt{n}} \to \underset{\text{(std normal)}}{\mathcal{N}(0,1)}$. *That is, for any bounded continuous function* f,

$$\lim_{n\to\infty} \boxed{\boldsymbol{E} f\Big(\frac{S_n}{\sqrt{n}}\Big)} = \boxed{\boldsymbol{E} f\big(\mathcal{N}(0,1)\big)}.$$

(Note $\operatorname{Var}\left(\frac{S_n}{\sqrt{n}}\right) = \frac{1}{n}\operatorname{Var}(S_n) = \frac{n}{n} = 1$

Note $E f(N(0,1)) = \int_{x\in\mathbb{R}} f(x)\,\underbrace{p(x)}_{\text{pdf}}\,dx$

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



Lecture 2 (1/21). Please enable video if you can.

Last time: Cts time markets

→ Black Scholes formula

- Replication
- Construct R part using same tricks from DTF

formula used CDF of Normal!

Theorem 4.7 (Central limit theorem)**.** $S_n/\sqrt{n} \to \mathcal{N}(0,1)$. *That is, for any bounded continuous function f,*

$$\boldsymbol{E} f\Big(\frac{S_n}{\sqrt{n}}\Big) = \boldsymbol{E} f\big(\mathcal{N}(0,1)\big)\,.$$

$$S_n = \sum_{1}^{n} X_k \qquad X_k \to \text{iid}, \quad EX_k = 0.\ EX_k^2 = 1$$

Q: In what sense does $\frac{S_n}{\sqrt{n}} \longrightarrow N(0,1)$?

(Choice conv

a.s. conv)

Opt ①: $\forall\, \omega \in \Omega$, want $\lim_{n\to\infty} \frac{S_n(\omega)}{\sqrt{n}} \longrightarrow N(0,1)(\omega)$

Mean sq conv.

Opt ②: Want $E\left|\frac{S_n}{\sqrt{n}} - N(0,1)\right|^2 \xrightarrow{n\to\infty} 0$

opt ③ Want the dist of $\underbrace{\frac{S_n}{\sqrt{n}}}_{\text{may be discrete RV's}}$ to conv $\longrightarrow$ $\underbrace{\text{dist}(N(0,1))}_{\text{cts.}}$

④ Take a "test function", $\underline{\underline{f}}$

$$E f\left(\frac{S_n}{\sqrt{n}}\right) \xrightarrow{\text{Want}} E f(\underline{N(0,1)})$$

for EVERY test function f.

(Analogy: Say f, g are 2 fnc (cts)

$$f : [0,1] \to \mathbb{R}$$
$$g : [0,1] \to \mathbb{R}$$

Q: Does $\int_0^1 f = \int_0^1 g \Rightarrow f = g$

(NO)

Q: Does $\int_0^1 f \cdot h = \int_0^1 g \cdot h$ $\forall$ test fn $h \Rightarrow \underline{f = g}$

Yes

(Reason: $\int_0^1 f\,h = \int_0^1 g\,h \quad \forall h$

$$\Rightarrow \int_0^1 (f-g)\,h = 0 \quad \forall h$$

Chose $h = \underline{f-g}$ $\Rightarrow$ $\underbrace{f = g}$!)

Let X be a random variable.

Definition 4.8. The *characteristic function* of X is defined by $\varphi_X(\lambda) = \boldsymbol{E} e^{i\lambda X}$.

$(i = \sqrt{-1}$

$e^{i\theta} = (\cos\theta + i\sin\theta)$

Definition 4.9. The *moment generating function* (MGF) of X is defined by $M_X(\lambda) = \boldsymbol{E} e^{\lambda X}$.

Example 4.10. If $X \sim N(0,1)$ then $\varphi_X(\lambda) = e^{-\lambda^2/2}$, and $\boxed{M_X(\lambda) = e^{\lambda^2/2}}$.

Note: X is a RV.

MGF of X (Notation M_X) is a fn.

domain $\cancel{\mathbb{R}}$ target $\mathbb{R}$.

subset of $\mathbb{R}$

Domain of char fn $\varphi_X = \mathbb{R}$. $(\because |e^{i\lambda X}| = 1)$

MGF of Normal.

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$X \sim N(0,1).$$

$$\varphi_X(\lambda) = \mathbb{E}\, e^{\lambda X} = \int_{\mathbb{R}} e^{\lambda x} \, \phi_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{\lambda x} \, e^{-x^2/2} \, \frac{dx}{\sqrt{2\pi}}.$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(x^2 - 2\lambda x + \lambda^2\right) + \lambda^2/2} \, \frac{dx}{\sqrt{2\pi}}$$

$$= e^{\lambda^2/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-\lambda)^2} \frac{dx}{\sqrt{2\pi}}$$

$y = x - \lambda$

$$= e^{\lambda^2/2} \underbrace{\int_{-\infty}^{\infty} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}}_{1}$$

$$= e^{\lambda^2/2}.$$

QED

(You check $\varphi_X(\lambda) = e^{-\lambda^2/2}$).

Theorem 4.11. $\boldsymbol{E}X^n = (-i)^n \varphi_X^{(n)}(0) = M_X^{(n)}(0)$. *In particular,* $\boldsymbol{E}X = -i\varphi'_X(0) = M'_X(0)$, *and* $\boldsymbol{E}X^2 = -\varphi''_X(0) = M''_X(0)$.

Remark 4.12. Here $f^{(n)}(0)$ denotes the n^{th} derivative of f at 0.

Reason:

$$\varphi_X(\lambda) = E\, e^{i\lambda X} \qquad \text{(def)}$$

$$\frac{d}{d\lambda}\varphi_X(\lambda) = \frac{d}{d\lambda}\left(E\, e^{i\lambda X}\right)$$

$$= E\left((iX)\cdot e^{i\lambda X}\right)$$

$$\Rightarrow \varphi_X'(0) = E\left((iX)\cdot 1\right) = i\,EX.$$

Repeat for higher mots.

Let X, Y be two random variables.

Theorem 4.13. *The following are equivalent.*

(1) *X and Y have the same distribution (PDF)*
(2) *X and Y have the same CDF.*
(3) *X and Y have the same characteristic function.*
(4) *X and Y have the same moment generating function.*

same notion of conv for CLT.

Theorem 4.14. *A sequence of random variables $(X_n) \to X$ (in distribution) if and only if $\boxed{\varphi_{X_n} \to \varphi_X}$ pointwise.*

Theorem 4.15. *A sequence of random variables $(X_n) \to X$ (in distribution) if and only if $M_{X_n} \to M_X$ pointwise.*

Remark 4.16. The proofs of Theorem 4.13–4.15 are beyond the scope of this course; we will use them without proof.

Lecture 3 (1/24). Please ENABLE VIDEO if you can

Last time: CF (characteristic fn)

$$\varphi_X(\lambda) = E\, e^{i\lambda X} \qquad (\varphi_X : \mathbb{R} \to \mathbb{C})$$

MGF: $M_X(\lambda) = E\, e^{\lambda X}$

$$EX = -i\,\varphi_X'(0) \qquad \left(\text{b/c} \quad \partial_\lambda \varphi_X(\lambda) = E\left(iX\underbrace{e^{i\lambda X}}\right) \Rightarrow \varphi_X'(0) = i\,EX \right)$$

$$EX^2 = -\varphi_X''(0)$$

Let X, Y be two random variables.

Theorem 4.13. *The following are equivalent.*

(1) X and Y have the same distribution (PDF)
(2) X and Y have the same CDF.
(3) X and Y have the same characteristic function.
(4) X and Y have the same moment generating function.

CDF of X: $F_X(x) = P(X \le x)$

i.e. $\forall \lambda \in \mathbb{R}$, $\varphi_{X_n}(\lambda) \to \varphi_X(\lambda)$

key

Theorem 4.14. *A sequence of random variables $(X_n) \to X$ (in distribution) if and only if $\varphi_{X_n} \to \varphi_X$ pointwise.*

Theorem 4.15. *A sequence of random variables $(X_n) \to X$ (in distribution) if and only if $M_{X_n} \to M_X$ pointwise.*

Need a certain assumption

Remark 4.16. The proofs of Theorem 4.13–4.15 are beyond the scope of this course; we will use them without proof.

Note: If X & Y have the same pdf f

then $\varphi_X(\lambda) = E e^{i\lambda X} = \int_{\mathbb{R}} e^{i\lambda x} f(x)\,dx$

$\varphi_Y(\lambda) = E e^{i\lambda Y} = \int_{\mathbb{R}} e^{i\lambda y} f(y)\,dy$

equal!

$\Rightarrow$ X & Y have the same C.F.

(III)' $\Rightarrow$ X & Y have the same MGF)

up

. Similarly

Rmk: Hard part of Thm 4.13 is showing same CF / MGF

$\Rightarrow$ same dist / CDF!

Note: We say $\varphi_{X_n} \longrightarrow \varphi_X$ pointwise if for every $\lambda \in \mathbb{R}$

$$\varphi_{X_n}(\lambda) \longrightarrow \varphi_X(\lambda).$$

Proof of Theorem 4.7. ~~4~~ ← CLT.

Thm 4.7: X_n iid. $\quad EX_n = 0, \quad EX_n^2 = 1$

$$S_n = \sum_{1}^{n} X_k .$$

std normal.

CLT: $$\frac{S_n}{\sqrt{n}} \xrightarrow{\text{dist}} N(0,1)$$

Strategy: Show $\mathrm{CF}\left(\frac{S_n}{\sqrt{n}}\right) \xrightarrow{\text{ptwise}} \mathrm{CF}(N(0,1))$

<u>Step 1</u>: Compute the form of the CF of X_k.

Know ① $EX_k = 0 = -i\varphi'_{X_k}(0) \Rightarrow \varphi'_{X_k}(0) = 0$

Know ② $EX_k^2 = 1 = -\varphi''_{X_k}(0) \Rightarrow \varphi''_{X_k}(0) = -1$

Know ③ $\varphi_{X_k}(0) = Ee^{i0X_k} = 1$

Expect $$\varphi_{X_k}(\lambda) = 1 + \underbrace{0}_{\varphi'_{X_k}(0)}\lambda \underbrace{- \frac{1}{2}}_{\varphi''_{X_k}(0)=-1}\lambda^2 + O(\lambda^3)$$

$$\boxed{\Rightarrow \varphi_{X_k}(\lambda) = 1 - \frac{\lambda^2}{2} + O(\lambda^3)}$$

Step 2: Find φ_{S_n}:

$$\varphi_{S_n}(\lambda) = E\, e^{i\lambda S_n} = E\, e^{i\lambda \sum_1^n X_k}$$

$$= E\left(\prod_{k=1}^{n} e^{i\lambda X_k}\right) \overset{\text{indep}}{=\!=} \prod_{k=1}^{n} E e^{i\lambda X_k}$$

$$= \prod_{k=1}^{n} \varphi_{X_k}(\lambda) = \left(1 - \frac{\lambda^2}{2} + O(\lambda^3)\right)^n$$

Step 3: $\varphi_{\left(\frac{S_n}{\sqrt{n}}\right)}(\lambda) = E\, e^{i\lambda S_n/\sqrt{n}}$

$$= E\, e^{i\left(\frac{\lambda}{\sqrt{n}}\right) S_n} = \varphi_{S_n}\left(\frac{\lambda}{\sqrt{n}}\right)$$

$$= \left[1 - \frac{\lambda^2}{2n} + O\left(\frac{\lambda^3}{n^{3/2}}\right)\right]^n$$

Step 4: Compute $\lim_{n\to\infty} \varphi_{\frac{S_n}{\sqrt{n}}}(\lambda)$

Know $\lim_{n\to\infty} \varphi_{\frac{S_n}{\sqrt{n}}}(\lambda) = \lim_{n\to\infty} \left(1 - \frac{\lambda^2}{2n} + O\left(\frac{\lambda^3}{n^{3/2}}\right)\right)^n$

$$= \lim_{n\to\infty} \exp\left(n \ln\left(1 - \underbrace{\frac{\boxed{\lambda^2}}{2n} + O\left(\frac{\lambda^3}{n^{3/2}}\right)}_{x}\right)\right)$$

$e^x = \exp(x)$

$$= \lim_{n\to\infty} \exp\left(\cancel{n}\left(-\frac{\lambda^2}{2\cancel{n}} + O\left(\frac{\lambda^3}{n^{3/2}}\right)\right)\right)$$

Know $\ln(1+x) \approx 0 + 1x + O(x^2)$

$\ln(1+x) \approx x$

$$= \exp\left(-\frac{\lambda^2}{2}\right)$$

$=$ char fn of $N(0,1)$ (evaluated at λ). QED.

5. Stochastic Processes.

5.1. Brownian motion.

- Discrete time: Simple Random Walk.
  - ▷ $X_n = \sum_1^n \xi_i$, where ξ_i's are i.i.d. $\boldsymbol{E}\xi_i = 0$, and $\operatorname{Range}(\xi_i) = \{\pm 1\}$.
- Continuous time: Brownian motion.
  - ▷ $Y_t = X_n + (t - n)\xi_{n+1}$ if $t \in [n, n+1)$.
  - ▷ Repeat more frequently: Flip a coin every ε seconds, and take a step of size $\sqrt{\varepsilon}$.
  - ▷ Rescale: $Y_t^\varepsilon = \sqrt{\varepsilon} Y_{t/\varepsilon}$. (Chose $\sqrt{\varepsilon}$ factor to ensure $\operatorname{Var}(Y_t^\varepsilon) \approx t$.)
  - ▷ Let $W_t = \lim_{\varepsilon \to 0} Y_t^\varepsilon$.

Definition 5.1 (Brownian motion)**.** The process W above is called a Brownian motion.

▷ Named after Robert Brown (a botanist).

▷ Definition is intuitive, but not as convenient to work with.

- If t, s are multiples of ε: $Y_t^\varepsilon - Y_s^\varepsilon \sim \sqrt{\varepsilon} \sum_{i=1}^{(t-s)/\varepsilon} \xi_i \xrightarrow{\varepsilon \to 0} \mathcal{N}(0, t-s)$.
- $Y_t^\varepsilon - Y_s^\varepsilon$ only uses coin tosses that are "after s", and so independent of Y_s^ε.

Definition 5.2. Brownian motion is a *continuous process* such that:

(1) $W_t - W_s \sim \mathcal{N}(0, t-s)$,
(2) $W_t - W_s$ is independent of $\mathcal{F}_s$.

Remark 5.3. We will define $\mathcal{F}_s$ shortly. Intuitively, $\mathcal{F}_s$ is the set of all events that are "known" at time s.

Lecture 4 (Jan 26) Please ENABLE VIDEO if you can.

Last time: CLT $\longrightarrow$ $X_n \longrightarrow$ iid, $EX_n=0$, $EX_n^2=1$

then $\dfrac{S_n}{\sqrt{n}} \longrightarrow \underbrace{N(0,1)}$

$\left(S_n = \sum_{k=0}^{n} X_k\right)$

n

Review from Prob

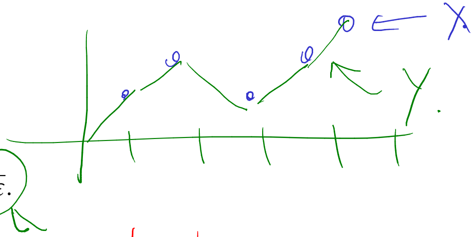
① Multivariate Normal → density, covariance matrix. etc

② Linear transformation of Normal → Normal.
(& can compute mean & cov)

(*) ③ limit of Normal → Normal (will say more in class).

5. Stochastic Processes.

5.1. Brownian motion.



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Wiener Process

cts time RW.

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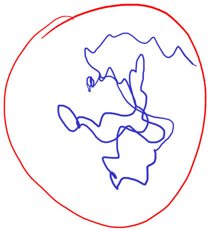


$Y_{\frac{t}{\varepsilon}}$

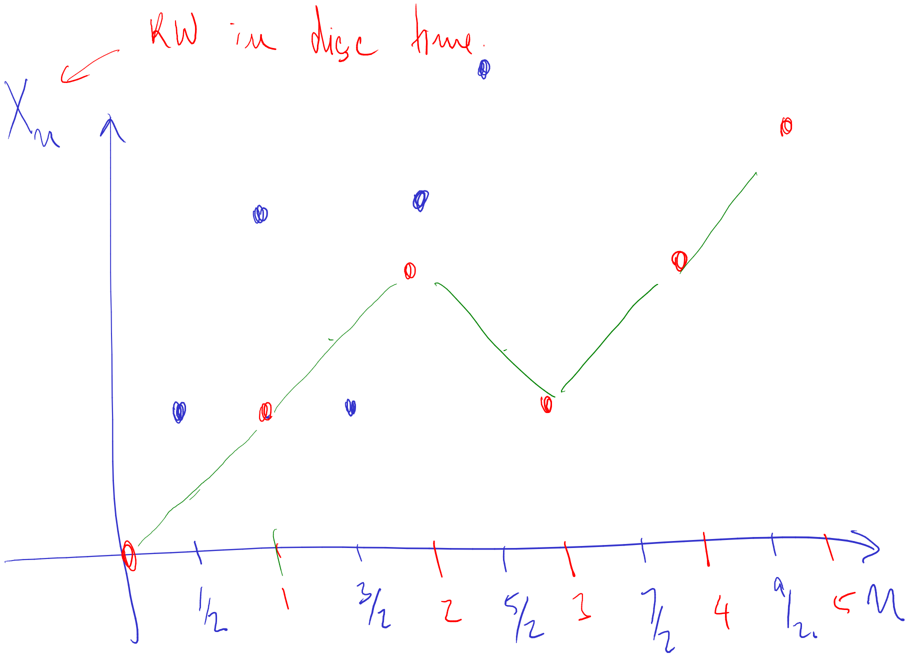
$Y_t \longrightarrow \lfloor t \rfloor$ steps of a RW (ctsly joined).

$Y_{t/\varepsilon} \longrightarrow \lfloor t/\varepsilon \rfloor$ steps of a RW (ctsly joined) step size 1.

$\sqrt{\varepsilon}\, Y_{t/\varepsilon} \longrightarrow \lfloor \frac{t}{\varepsilon} \rfloor$ steps of a RW (ctsly joined).
step size $\sqrt{\varepsilon}$

$\xi_k \to$ iid.

$$X_n = \sum_{k=1}^{n} \xi_k$$



Want RW in cts time

Want to keep variance at time 1 constant

(1) Flip coins every second: Var at time 1 = 1

(step size 1)

(2) Flip coins every $\frac{1}{2}$ secs: Var at time $1 = 1+1 = 2$.

step size: $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{\sqrt{2}}$ Var at time $1 = 1_n$

(4) 4 " " $\frac{1}{n}$ secs: Var at time $1 = 2$.

step size $\frac{1}{2^{n/2}}$ 2^n $\longrightarrow$ " " " $1 = 1$.

To keep variance constant: decrease size of the step as we decrease the interval between coin flips.

$$\xi_1 = \begin{cases} a & \text{prob } \frac{1}{2} \\ -a & \text{prob } \frac{1}{2}. \end{cases}$$

(a = step size)

$\mathrm{Var}(\xi_1) = a^2$. Want Var after 2 steps $= 1 \Leftrightarrow 2a^2 = 1 \Leftrightarrow a = \frac{1}{\sqrt{2}}$

Let $\underset{\sim}{W_t} = \lim_{\varepsilon \to 0} \sqrt{\varepsilon}\, Y_{t/\varepsilon} = \lim_{\varepsilon \to 0} Y^{\varepsilon}_t .$

Q: Does this lim exist? $\Rightarrow$ Yes (Hard)

Q: Can we say something about the lim? $\Rightarrow$ Yes (CLT!)

- If t, s are multiples of ε: $\boxed{Y_t^\varepsilon - Y_s^\varepsilon \sim \sqrt{\varepsilon} \sum_{i=1}^{(t-s)/\varepsilon} \xi_i \xrightarrow{\varepsilon \to 0} \mathcal{N}(0, t-s).}$
- $Y_t^\varepsilon - Y_s^\varepsilon$ only uses coin tosses that are "after s", and so independent of Y_s^ε.

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$$W_t = \lim_{\varepsilon \to 0} \sqrt{\varepsilon}\, Y_{t/\varepsilon}.$$

s, t mult of ε. Pretend $W_t \approx \sqrt{\varepsilon}\, Y_{t/\varepsilon}$

$W_s \approx \sqrt{\varepsilon}\, Y_{s/\varepsilon}$

$$W_t - W_s \approx \sqrt{\varepsilon}\left(Y_{t/\varepsilon} - Y_{s/\varepsilon} \right) = \sqrt{\varepsilon}\left(X_{t/\varepsilon} - X_{s/\varepsilon} \right)$$

$$= \sqrt{\varepsilon}\left(\sum_{k=1}^{t/\varepsilon} \xi_k - \sum_{k=1}^{s/\varepsilon} \xi_k\right)$$

$$= \sqrt{\varepsilon}\underbrace{\sum_{k=s/\varepsilon}^{t/\varepsilon} \xi_k}_{\text{sum of } \frac{t-s}{\varepsilon} \text{ iid RV's}} \xrightarrow{\varepsilon \to 0}$$

$$= \sqrt{t-s}\,\frac{1}{\left(\frac{t-s}{\varepsilon}\right)^{1/2}} \sum_{k=s/\varepsilon}^{t/\varepsilon} \xi_k .$$

$$\text{CLT} \xrightarrow[\varepsilon \to 0]{} N(0,1).$$

$$\longrightarrow \sqrt{t-s}\; N(0,1) = N(0, t-s).$$

Bottom line: Expect $W_t - W_s \sim N(0, t-s)$.

Lecture 5 (Jan 28)

Please ENABLE VIDEO If you can.

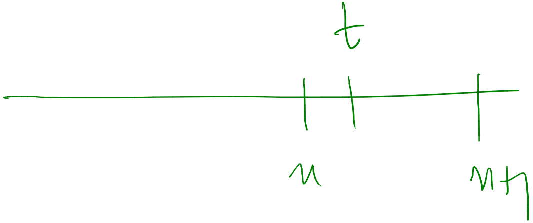
Last time → Construct B.M.

Brownian Motion → cts time R.W.

R.W. : $X_n = \sum_{1}^{k} \xi_k$, $\xi_k \to$ iid, $E\xi_k = 0$, $E\xi_k^2 = 1$

↪ $X_{n+1} = X_n + \xi_{n+1}$.

$$Y_t = Y_n + (t-n)\,\xi_{n+1} \qquad t \in [n, n+1)$$



Flip coin every ε seconds

Let $Y^{\varepsilon}_t = \sqrt{\varepsilon}\, Y_{t/\varepsilon}$ (RW with step size $\sqrt{\varepsilon}$ & coin flips occurring every ε seconds)

B. M. $\rightarrow$ cts time RW $\rightarrow$ send $\varepsilon \rightarrow 0$

Define $W_t = \lim_{\varepsilon \to 0} Y^{\varepsilon}_t = \lim_{\varepsilon \to 0} \sqrt{\varepsilon}\, Y_{t/\varepsilon}$.

- If t, s are multiples of ε: $Y_t^\varepsilon - Y_s^\varepsilon \sim \sqrt{\varepsilon} \sum_{i=1}^{(t-s)/\varepsilon} \xi_i \xrightarrow{\varepsilon \to 0} \mathcal{N}(0, t-s)$. (CLT, last time)
- $Y_t^\varepsilon - Y_s^\varepsilon$ only uses coin tosses that are "after s", and so independent of Y_s^ε.

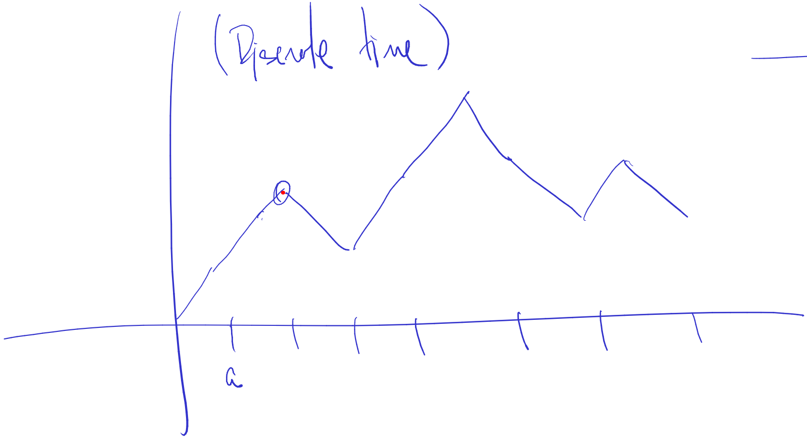
Definition 5.2. Brownian motion is a *continuous process* such that: $W_0 = 0$

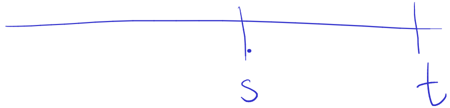
(1) $W_t - W_s \sim \mathcal{N}(0, t-s)$,
(2) $W_t - W_s$ is independent of $\mathcal{F}_s$.

Remark 5.3. We will define $\mathcal{F}_s$ shortly. Intuitively, $\mathcal{F}_s$ is the set of all events that are "known" at time s.

Pickup

(Discrete time)





For any $s < t$ expect the BM to change direction infinitely often between times s & t.

Also ~~know~~ Expect B.M to be a cts fn of time.

Claim: BM is always a cts fn of time

& With Prob 1, BM is not diff anywhere (wrt time)

Finance : Standard model for stock price

(Geometric B.M.) $$S_t = S_0 \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W_t \right)$$

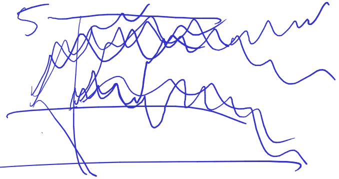
$\alpha \to$ mean return rate

$\sigma \to$ volatility

$\boxed{W \to \text{Brownian motion!!}}$

5.2. Sample space, measure, and filtration.

- Discrete time: Sample space $\Omega = \{(\omega_1, \dots, \omega_N)\}$. ω_i = outcome of i^{th} coin toss $\}$.
- View $(\omega_1, \dots, \omega_N)$ as the trajectory of a random walk.
- Continuous time: Sample space $\Omega = C([0, \infty))$ (space of continuous functions).
  - ▷ It's infinite. No probability mass function!
  - ▷ Mathematically impossible to define $\boldsymbol{P}(A)$ for *all* $A \subseteq \Omega$.



→ PMF

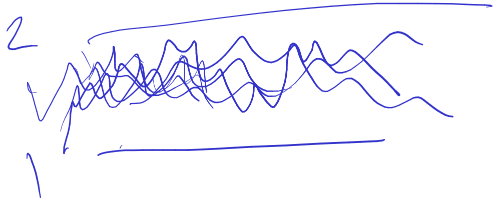
$p : \Omega \to \mathbb{R}$, $\boxed{\sum_{\omega \in \Omega} p(\omega) = 1}$

Def $P(A) = \sum_{\omega \in A} p(\omega)$

Discrete time

$\Omega = \{(\omega_1, \dots, \omega_N) \mid \omega_i \to i^{\text{th}}$ coin toss$\}$ → can't easily generalize.

→ $\mathcal{F}_n$ = all events that can be observed using only coin tosses before n.



$$X_{n+1} = X_n + \underbrace{\omega_{n+1}}_{n+1^{\text{th}}} \text{ coin toss.} \qquad (RW)$$

$$(\omega_1, \dots \omega_N) \longrightarrow \text{tells me the trajectory of the RW}$$

Cts time: Sample space $= \{\text{paths BM can take}\}$

$= \{\text{all cts fns, domain } [0,\infty) \text{ \& target } \mathbb{R}\}$

$= C([0,\infty))$ (intuition

- Restrict our attention to $\mathcal{G}$, a subset of some sets $A \subseteq \Omega$, on which $\boldsymbol{P}$ can be defined.
  - ▷ $\mathcal{G}$ is a σ-algebra. (Closed countable under unions, complements, intersections.)
- $\boldsymbol{P}$ is called a *probability measure* on $(\Omega, \mathcal{G})$ if:
  - ▷ $\boldsymbol{P}\colon \mathcal{G} \to [0,1]$, $\boldsymbol{P}(\emptyset) = 0$, $\boldsymbol{P}(\Omega) = 1$.
  - ▷ $\boldsymbol{P}(A \cup B) = \boldsymbol{P}(A) + \boldsymbol{P}(B)$ if $A, B \in \mathcal{G}$ are disjoint.
  - ▷ ~~If $A_n \in \mathcal{G}$, $\boldsymbol{P}\left(\bigcup_1^\infty A_n\right) = \lim_{n\to\infty} \boldsymbol{P}(A_n)$.~~
- Random variables are *measurable* functions of the sample space:
  - ▷ Require $\{X \in A\} \in \mathcal{G}$ for every "nice" $A \subseteq \mathbb{R}$.
  - ▷ E.g. $\{X = 1\} \in \mathcal{G}$, $\{X > 5\} \in \mathcal{G}$, $\{X \in [3,4)\} \in \mathcal{G}$, etc.
  - ▷ Recall $\{X \in A\} = \{\omega \in \Omega \mid X(\omega) \in A\}$.

elts of $\mathcal{G}$.

If $A_1, A_2, \dots$ are ∞'ly many disj ~~sets~~ events then $P\left(\bigcup_{n=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$

$\Omega \Rightarrow$ Sample space

$\mathcal{G}$ is a σ-algebra on Ω if

① $\mathcal{G}$ is a non-empty collection of subsets of Ω

② $\phi \in \mathcal{G}$, $\Omega \in \mathcal{G}$.

③ If $A, B \in \mathcal{G}$, then A^c, $A \cup B$, $A \cap B \in \mathcal{G}$.

④ If $A_1, A_2 \dots \in \mathcal{G}$ then $\bigcup_{1}^{\infty} A_k \in \mathcal{G}$.

Say $X : \Omega \longrightarrow \mathbb{R}$.

Q: $P(X > 0) = P\left(\overbrace{\{\omega \in \Omega \mid X(\omega) > 0\}} \right)$

$= P\left(\{X > 0\} \right)$

Lecture 8 (2/4). Please ENABLE VIDEO if you can

<u>Last time</u>: Cond exp. → NO explicit formula

→ SAME properties as in the disc case

(examples and w/o being a bit harder)

Definition 5.5. $\boxed{\boldsymbol{E}_t X}$ is the unique *random variable* such that:

(1) $\boldsymbol{E}_t X$ is $\mathcal{F}_t$-measurable.

(2) For every $A \in \mathcal{F}_t$, $\int_A \boldsymbol{E}_t X \, d\boldsymbol{P} = \int_A X \, d\boldsymbol{P}$

(i.e. $E(1_A E_t X) = E(1_A X)$)

Remark 5.6. Choosing $A = \Omega$ implies $\boldsymbol{E}(\boldsymbol{E}_t X) = \boldsymbol{E} X$.

Proposition 5.7 (Useful properties of conditional expectation)**.**

(1) If $\alpha, \beta \in \mathbb{R}$ are constants, X, Y, random variables $\boldsymbol{E}_t(\alpha X + \beta Y) = \alpha \boldsymbol{E}_t X + \beta \boldsymbol{E}_t Y$.

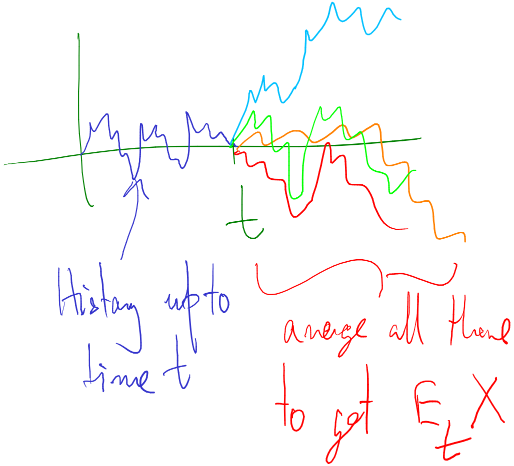
(2) If $X \geqslant 0$, then $\boldsymbol{E}_t X \geqslant 0$. Equality holds if and only if $X = 0$ almost surely.

(3) (Tower property) *If $0 \leqslant s \leqslant t$, then $\boldsymbol{E}_s(\boldsymbol{E}_t X) = \boldsymbol{E}_s X$.*

(4) If X is $\mathcal{F}_t$ measurable, and Y is any random variable, then $\boldsymbol{E}_t(XY) = X \boldsymbol{E}_t Y$.

(5) If X is $\mathcal{F}_t$ measurable, then $\boldsymbol{E}_t X = X$ (follows by choosing $Y = 1$ above).

(6) If Y is independent *of $\mathcal{F}_t$, then $\boldsymbol{E}_t Y = \boldsymbol{E} Y$.*



Remark 5.8. These properties are exactly the same as in discrete time.

Lemma 5.9 (Independence Lemma)**.** *If X is $\mathcal{F}_t$ measurable, Y is independent of $\mathcal{F}_t$, and $f = f(x,y) \colon \mathbb{R}^2 \to \mathbb{R}$ is any function, then*

$$\cancel{\boldsymbol{E}_t f(X,Y) = g(X)\,, \quad \textit{where} \quad g(y) = \boldsymbol{E} f(X,y)}.$$

Remark 5.10. If p_Y is the PDF of Y, then $\boldsymbol{E}_t f(X,Y) = \int_{\mathbb{R}} f(X,y)\, p_Y(y)\, dy$.

Thm $E_t f(X, Y) =$ "average Y & leave X alone"

$= g(X)$ where $g(x) = E f(x, \overset{\text{RV}}{Y})$

Example 5.11. If X, Y are two independent standard normal random variables, find $\boldsymbol{E}e^{iXY}$.

Option 1: X & Y ind normal $\Rightarrow$ Joint PDF of (X, Y) is

$$\frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

$$\Rightarrow E e^{iXY} = \int_{\mathbb{R}^2} e^{ixy} \cdot e^{-(x^2+y^2)/2} \frac{dx\,dy}{2\pi}.$$

& compute this integral

Option 2: Nice trick using the indep Lemma.

X, Y indep.

Let $\mathcal{F} = \sigma(X) =$ all events that can be observed using the RV X

i.e. $\{X>0\} \in \sigma(X)$. $\{X \in [1,2]\} \in \sigma(X)$

Obs: X is meas wrt $\sigma(X)$ & Y is ind of $\sigma(X)$ $\Big\} \to$ Use indep lemma!!

Notation $E_t(Z) = E(Z \mid \mathcal{F}_t)$

Compute: $E(e^{iXY}) \overset{\text{tower}}{=} E\left(E\left(e^{iXY} \mid \sigma(X)\right)\right)$

Notation

$= E\left(E\left(e^{iXY} \mid X\right)\right)$

Notation

By the indep lemma,

$$E\left(e^{iXY} \,\middle|\, X\right) = \underline{\text{avg } Y}, \ \overbrace{\text{leave } X \text{ alone}}$$

$$= g(X) \quad \text{where } g(x) = E\left(e^{i\underline{x}Y}\right)$$

$$= \text{char fn of std norml}$$

$$= e^{-x^2/2}$$

$$= e^{-X^2/2}$$

$$\Rightarrow E\left(e^{iXY} \,\middle|\, X\right) = e^{-X^2/2}$$

$$\Rightarrow E e^{iXY} = E\left(E\left(e^{iXY} \,\middle|\, X\right)\right) = E e^{-X^2/2}$$

$$= \int_{-\infty}^{\infty} e^{-x^2/2} \quad e^{-x^2/2} \, \frac{dx}{\sqrt{2\pi}}$$

$$= \underbrace{\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\cdot\left(\frac{1}{\sqrt{2}}\right)^2}} \, \frac{dx}{\sqrt{2\pi\left(\frac{1}{\sqrt{2}}\right)^2}}}_{1} \left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}.$$

M_t is $\mathcal{F}_t$ meas $\forall t$

5.4. **Martingales.**

Definition 5.12. An adapted process M is a martingale if for every $0 \leqslant s \leqslant t$, we have $\boldsymbol{E}_s M_t = \boxed{M_s}$.

Remark 5.13. As with discrete time, a martingale is a fair game: stopping based on information available today will not change your expected return.

Proposition 5.14. $\boxed{\textit{Brownian motion is a martingale.}}$

Proof.

(Analog in disc time : ξ_n iid, $E\xi_n = 0$, Set $X_{n+1} = X_n + \xi_{n+1}$

i.e $X_n = \sum_1^n \xi_k$

$X \to$ discr time RW.

Know X is a mg (from 370).

Check BM is a mg?

Know $W_0 = 0$, $W_t - W_s \sim N(0, t-s)$

& $W_t - W_s$ is ind of $\mathcal{F}_s$.

NTS. BM is a mg

i.e. NTS $\forall\ 0 \le s \le t$, $E_s W_t = W_s$

Pf: $E_s W_t = E_s(W_t - W_s + W_s)$

$= E_s(W_t - W_s) + E_s W_s$

$= E(W_t - W_s) + W_s$

$= 0 + W_s$ $= \boxed{W_s}$

$\Rightarrow$ B.M. is a mg.

Q: Is W_t^2 a mg? NO

Is W_t^3 a mg? NO

On HW: Compute $E_s W_t^3$

4/15 Lecture ?? (maybe 25?)

PLEASE ENABLE VIDEO IF YOU CAN

Need to be able to compute $\tilde{E}_t(D_T V_T)$

No nice formula

RNP Formula: $V_t = \frac{1}{D_t} \tilde{E}_t(D_T V_T)$

9.3. **Constructing Risk Neutral Measures.** Suppose the market has only one stock whose price process satisfies

Bank pays interest rate R_t

$$dS_t = \alpha_t S_t \, dt + \sigma_t S_t \, dW_t \, .$$

(α, σ both processes)

Theorem 9.17. *The (unique) risk neutral measure is given by $d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}$, where*

$$Z_T = \exp\Big(-\int_0^T \theta_t \, dW_t - \frac{1}{2}\int_0^T \theta_t^2 \, dt\Big) , \qquad \boxed{\theta_t = \frac{\alpha_t - R_t}{\sigma_t} .} \leftarrow \text{Market price of risk.}$$

Proposition 9.18. *The stock price satisfies*

$$\longrightarrow dS_t = R_t S_t \, dt + \sigma_t S_t \, d\tilde{W} ,$$

where $\tilde{W}$ is a Brownian motion under the risk neutral measure.

Last time: $d\tilde{W} = \theta \, dt + dW$

Found $d\tilde{P}$ using Girsanov ($\Rightarrow$ Under $\tilde{P}$, $\tilde{W}$ is a BM!)

Pf: Know $dS_t = \alpha_t S_t \, dt + \sigma_t S_t \, dW_t$

$= \alpha_t S_t \, dt + \sigma_t S_t \left(-\theta \, dt + d\tilde{W}\right)$

$$= \cancel{\alpha S\,dt} + \sigma S\left(\frac{R-\cancel{\alpha}}{\sigma}\,dt + d\tilde{W}\right)$$

$$= R_t S_t\,dt + \sigma_t S_t\,d\tilde{W}$$

Useful under P: $\quad dS = \alpha_t S_t\,dt + \sigma_t S_t\,dW$

Useful under $\tilde{P}$: $\quad dS = R_t S_t\,dt + \sigma_t S_t\,d\tilde{W}$

9.4. **Black Scholes Formula revisited.**

- Suppose the interest rate $R_t = r$ (is constant in time).
- Suppose the price of the stock is a GBM(α, σ) (both α, σ are constant in time).

$dS = \alpha S\, dt + \sigma S\, dW$

Theorem 9.19. *Consider a security that pays $V_T = g(S_T)$ at maturity time T. The arbitrage free price of this security at any time $t \leqslant T$ is given by $f(t, S_t)$, where*

$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\Big(x \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\, y\Big)\Big) \frac{e^{-y^2/2}\,dy}{\sqrt{2\pi}}, \qquad \tau = T - t. \tag{7.4}$$

Remark 9.20. This proves Proposition 7.8.

Pf: Use RNM. Know under $\tilde{P}$, $dS_t = r S\, dt + \sigma S\, d\tilde{W}_t$

($\tilde{W}$ is a BM under $\tilde{P}$) $\Rightarrow S = \text{GBM}(r, \sigma)$ under $\tilde{P}$.

$$\Rightarrow S_t = S_0 \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)t + \sigma \tilde{W}_t\Big)$$

$$S_T = S_0 \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)T + \sigma \tilde{W}_T\Big)$$

$$\Rightarrow \quad \frac{S_T}{S_t} = \exp\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\left(\tilde{W}_T - \tilde{W}_t\right)\right)$$

$$\Rightarrow \quad \underline{S_T} = \underline{S_t}\, \underbrace{\exp\left(\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\left(\tilde{W}_T - \tilde{W}_t\right)\right)}$$

(2) RNP Formula:

$$V_t = \text{AFP at time } t = \frac{1}{D_t}\tilde{E}_t\left(D_T V_T\right) \qquad \left(D_t = e^{-rt}\right)$$

$$= e^{-r(T-t)}\tilde{E}_t V_T = e^{-r(T-t)}\tilde{E}_t\, g(\underline{S_T})$$

$$= e^{-r\tau}\,\tilde{E}_t\, g\Big(\underbrace{S_t}_{\mathcal{F}_t\text{-meas}} \exp\Big[\Big(r-\frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\,\underbrace{\frac{(\tilde{W}_T-\tilde{W}_t)}{\sqrt{\tau}}}_{\text{ind of } \mathcal{F}_t \;\sim N(0,1)}\Big]\Big)$$

derived formula!

$$\boxed{V_t \overset{\text{indep lemma}}{=} e^{-r\tau}\int_{y=-\infty}^{\infty} g\Big(S_t \exp\Big[\Big(r-\frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\,y\Big)\,\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy}$$

Theorem 9.21 (Black Scholes Formula). *The arbitrage free price of a European call with strike K and maturity T is given by:*

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)) \tag{7.5}$$

where

$$d_\pm(\tau,x) \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big), \tag{7.6}$$

and

$$N(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy, \tag{7.7}$$

is the CDF of a standard normal variable.

Remark 9.22. This proves Corollary 7.9.

Pf: AFP of call at time $t = V_t = c(t, S_t) = e^{-r\tau}\tilde{E}_t(S_T - K)^+$

By 9.19: $c(t,x) = e^{-r\tau}\int_{-\infty}^{\infty}\Big(\underbrace{x\exp\Big[\Big(r-\frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\,y\Big] - K}_{\text{when is this } >0\,?}\Big)^+ e^{-y^2/2}\,\frac{dy}{\sqrt{2\pi}}$

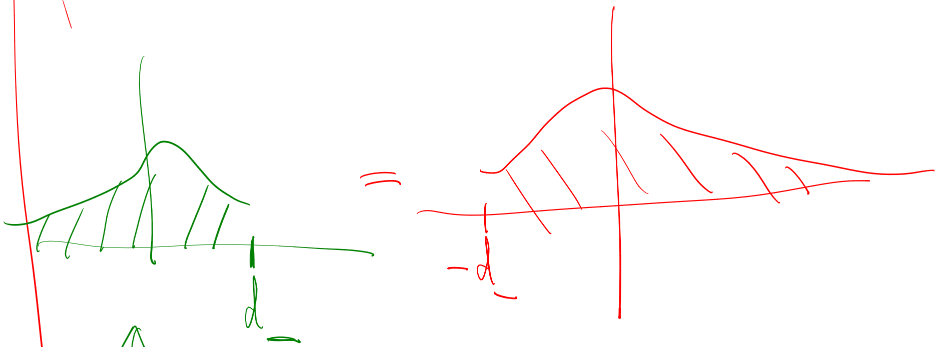
Solve $x\, e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\, y} = K$

$$\iff \left(r-\frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\, y = \ln\left(\frac{K}{x}\right)$$

$$\iff y = \frac{-1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(r-\frac{\sigma^2}{2}\right)\tau\right) = -d_-$$

$$\Rightarrow c(t,x) = e^{-r\tau}\int_{-d_-}^{\infty}\left(x\, e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\, y} - K\right)^{\cancel{+}} e^{-\frac{y^2}{2}}\,\frac{dy}{\sqrt{2\pi}}$$

$$= \underbrace{\int_{-d_-}^{\infty} x\, e^{\left(r-\frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\, y - r\tau} \cdot e^{-y^2/2}\, \frac{dy}{\sqrt{2\pi}}}_{\text{Simplify}} - K e^{-r\tau} \underbrace{\int_{-d_-}^{\infty} e^{-y^2/2}\, \frac{dy}{\sqrt{2\pi}}}_{N(d_-)}$$



$P(N(0,1) \leq d_-) = N(d_-)$

a

$$= -Ke^{-r\tau}N(d_-) + \int_{-d_-}^{\infty} x e^{-\frac{1}{2}\left(\sigma^2\tau - 2\sigma\sqrt{\tau}\,y + y^2\right)} \frac{dy}{\sqrt{2\pi}}$$

$$= -Ke^{-r\tau}N(d_-) + x\int_{-d_-}^{\infty} e^{-\frac{1}{2}\left(y - \sigma\sqrt{\tau}\right)^2} \frac{dy}{\sqrt{2\pi}}$$

Put $z = y - \sigma\sqrt{\tau}$

$dz = dy$

$$= \quad '' \quad + x\int_{\underbrace{-(d_- + \sigma\sqrt{\tau})}_{d_+}}^{\infty} e^{-z^2/2} \frac{dz}{\sqrt{2\pi}} = xN(d_+) - Ke^{-r\tau}N(d_-)$$

//

9.5. **The Martingale Representation Theorem.**

Theorem 9.23. *If M_t is a square integrable martingale with respect to the Brownian filtration, then there exists a* predictable process *D such that $\boldsymbol{E}\int_0^t D_s^2\,ds < \infty$ and*

$$M_t = M_0 + \int_0^t D_s\,dW_s\,.$$

Remark 9.24. A *square integrable* martingale is a martingale for which $\boldsymbol{E}M_t^2 < \infty$ for all t.

Remark 9.25. For our purposes, think of a *predictable process* as a left continuous and adapted process.

Theorem 9.26. *Consider the one stock market form Theorem 9.17.*

1. *Any $\tilde{\boldsymbol{P}}$ martingale is the discounted wealth of a self financing portfolio (i.e. converse of Theorem 9.5 holds)*
2. *Any security with an $\mathcal{F}_T$-measurable payoff is replicable, and so Theorem 9.7 holds for any $\mathcal{F}_T$-measurable function V_T.*
3. *The risk neutral measure is unique.*

370 $\to$ Existence of RNM $\iff$ No arb

Exist & Uniq $\iff$ No arb & completeness

Lecture ?? (4/18).

Please Enable Video if you can.

Last time: Know ① Compute RNM (1 stock 1 Bank)

② Know RNP formula & how to compute

(Main rule: Write all asset / security prices in terms of $\tilde{W}$ & use the fact that $\tilde{W}$ is a B.M. under $\tilde{P}$).

9.5. The Martingale Representation Theorem.

(i.e M is a mg & $\forall t$, $E M_t^2 < \infty$

Theorem 9.23. *If M_t is a square integrable martingale with respect to the Brownian filtration, then there exists a* predictable process *D such that $\boldsymbol{E}\int_0^t D_s^2\,ds < \infty$ and*

$$M_t = M_0 + \int_0^t D_s\,dW_s\,.$$

(M_0 is not random).

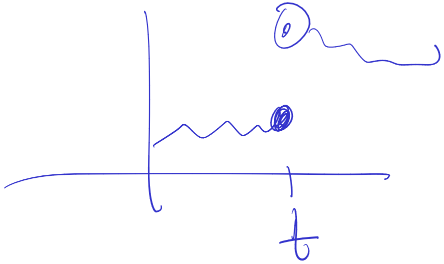
Remark 9.24. A *square integrable* martingale is a martingale for which $\boldsymbol{E}M_t^2 < \infty$ for all t.

Remark 9.25. For our purposes, think of a *predictable process* as a left continuous and adapted process.

370: A predictable process is a process + X_n is $\mathcal{F}_{n-1}$ meas.

420: A predictable process is one for which

$$\lim_{s \to t^-} X_s = X_t$$



Note: We know Ito int wrt BM are mg's. Mg rep $\Rightarrow$ the converse.

Theorem 9.26. *Consider the one stock market form Theorem 9.17.* $dS_t = \alpha_t S_t dt + \sigma_t S_t dW_t$

(1) *Any $\tilde{\boldsymbol{P}}$ martingale is the discounted wealth of a self financing portfolio (i.e. converse of Theorem 9.5 holds)*

(2) *Any security with an $\mathcal{F}_T$-measurable payoff is replicable, and so Theorem 9.7 holds for any $\mathcal{F}_T$-measurable function V_T.*

(3) *The risk neutral measure is unique.*

RNP

370: Dis wealth is mg $\iff$ self fin.

420: Thm 9.5: Self fin $\implies$ Disc wealth is a mg.

Mg rep thm $\implies$ converse.

Pf of (1) Assume X_t is process & $\boxed{D_t X_t \text{ is a } \tilde{P} \text{ mg}}$.

NTS X_t = wealth of a self fin port.

$\Longleftrightarrow$ i.e. NTS $\exists$ an $\mathcal{F}_t$ adapted process Δ_t $\ni$

$$dX_t = \Delta_t \, dS_t + R_t (X_t - \Delta_t S_t) \, dt \quad \longleftarrow \text{Want.}$$

($R_t \to$ interest rate at time t, $D_t = \exp\left(-\int_0^t R_s \, ds\right)$

$\Longleftrightarrow dD_t = -R_t D_t \, dt$ & $D_0 = 1$)

① Know $D_t X_t$ is a $\tilde{P}$ mg.

Use mg rep thm (under $\tilde{P}$) to generate

$$\exists \, \Gamma_t \text{ (adapted) } \ni D_t X_t = D_0 X_0 + \int_0^t \Gamma_s \, d\tilde{W}_s.$$

$$\boxed{\Longleftrightarrow \quad d(D_t X_t) = \underbrace{\Gamma_t}\, d\widetilde{W}_t \quad \longleftarrow \text{Have} \;--\; (*)}$$

② Want

$$dX_t = \Delta_t \, dS_t + R_t (X_t - \Delta_t S_t)\, dt$$

$$= \Delta_t \big(\underbrace{\alpha_t} S_t \, dt + \sigma_t S_t \underbrace{dW_t}_{\text{make } d\widetilde{W}} \big) + R (X_t - \Delta_t S_t)\, dt$$

$$= \Delta_t \big(\underbrace{R_t} S_t \, dt + \sigma_t S_t \, d\widetilde{W} \big) + R_t (X_t - \Delta_t S_t)\, dt$$

$$\boxed{\Rightarrow dX_t = \Delta_t \sigma_t S_t \, d\widetilde{W} + R_t X_t \, dt} \longleftarrow \text{Want.}$$

Scratch: Assume what we want (i.e assume $\underbrace{dX_t = \Delta_t S_t d\tilde{W} + R_t X_t dt}$)

& compute $d(D_t X_t)$:

$$\boxed{d(D_t X_t) = D_t \, dX_t + X_t \, dD_t + d[X, D]_t} \qquad 0$$

$$= D_t(\Delta_t \sigma_t S_t d\tilde{W} + \cancel{R_t X_t dt}) + X_t(\cancel{-R_t D_t dt})$$

$$\boxed{\Rightarrow d(D X_t) = \underbrace{D_t \Delta_t \sigma_t S_t}\, d\tilde{W}. \qquad **}$$

Actual proof: Know $\exists\, \Gamma_t$ (mg rep thm) $\ni$ $d(D_t X_t) = \Gamma_t \, d\tilde{W}$.

Choose $\Delta_t = \dfrac{\Gamma_t}{D_t \sigma_t S_t}$

Work backward through the above calculation (Scratch in green)

& get $dX_t = \Delta_t \, dS_t + R_t (X_t - \Delta_t S_t)\, dt$

$\Rightarrow X$ = wealth of a self fin port! QED.

Pf of ②: Say a sec pays V_T at time T

(V_T is $\mathcal{F}_T$ - meas)

NTS: Security is replicable.

Pf: Define X_t by $\underline{D_t X_t} = \underbrace{\tilde{E}_t(D_T V_T)}$

i.e. let $X_t = \frac{1}{D_t} \tilde{E}_t(D_T V_T)$.

<u>NOTE</u> $D_t X_t$ is a $\tilde{P}$- mg!

$$(\because \ \tilde{E}_s(D_t X_t) = \tilde{E}_s \hat{E}_t (D_T V_T) \overset{\text{tower}}{=} \tilde{E}_s (D_T V_T) = D_s X_s)$$

By thm part ① $\Rightarrow$ $D_t X_t$ is the disc wealth of a self fin port

$\Rightarrow$ X_t is the wealth of a self fin port.

Also $\underline{\underline{X_T}} = \dfrac{1}{D_T} \tilde{E}_T (D_T \underline{\underline{V_T}}) = \underline{\underline{V_T}}$

$\Rightarrow$ Sec is replicable QED.