21-370 Discrete Time Finance: Midterm 2.

2021-11-03

- *This is an open book test. You may use your notes, homework solutions, books, and/or online resources (including software) while doing this exam.*
- *You may* not*, however, seek or receive assistance from a live human during the exam. This includes in person assistance, instant messaging, and/or posting on online forums / discussion boards. (Searching discussion boards is OK, though.)*
- *Unsupported (or unreadable) answers will not receive credit. Show your work and explain your solution.*
- *You must record yourself (audio, video and screen) and share it with me as instructed by email.*
- *Late submissions will not be accepted. Please ensure you allow yourself ample time to scan your exam, otherwise you will get zero credit.*
- *You have 60 minutes. The exam has a total of 3 questions and 40 points.*
- *The questions are roughly in order of difficulty; though depending on your intuition, some of you may feel differently. Nevertheless, I suggest looking over the entire exam before starting. Good luck* ☺.

10 1. Consider the N period binomial model with $N = 3$ and interest rate $r = 1/8$. At time n we flip a coin (that is independent of previous coin flips) and lands heads with probability 0.9, and tails with probability 0.1. Let S_n denote the stock price at time n, and suppose $S_0 = \$20$. When the coin flips heads the stock price increases by 50% (i.e. $S_{n+1} = \frac{3}{2}S_n$.) When the coin flips tails the stock price decreases by 25% (i.e. $S_{n+1} = \frac{3}{4}S_n$.) Consider an American option with intrinsic value G, where

$$G_n = \begin{cases} 0 & n = 0\,, \\ \max\{0, S_n - (1+r)S_{n-1}\} & n > 0\,. \end{cases}$$

Let V_n denote the arbitrage free price of this option, and σ^* be the minimal optimal exercise policy for this option.

(a) Suppose $n = 2$, the first coin toss is tails and the second coin toss is heads. In this case find the arbitrage free price, and determine whether or not $n \leqslant \sigma^*$.

(b) Suppose $n = 2$, the first coin toss is heads and the second coin toss is tails. In this case find the arbitrage free price, and determine whether or not $n \leqslant \sigma^*$.

10 2. Consider the N period binomial model with $N = 5$, with interest rate $r = 1/2$. Let S_n denote the stock price at time n. At every time period, we flip a fair coin (independent of previous coins). If the coin lands heads, we set $S_{n+1} = 2S_n$. If the coin lands tails, we set $S_{n+1} = \frac{1}{2}S_n$. The stock price is initially \$8.

Let $\tau = \min\{n \leqslant N \mid S_n \notin (2, 32)\}$, and let $\sigma = \min\{N, \tau\}$. A rebate option matures at time σ and pays the holder \$6 if $S_\sigma \geqslant 32$, pays \$3 if $S_\sigma \leqslant 2$, and pays nothing otherwise. Find the arbitrage free price of this option at time $n = 1$. Also find the number of shares held in the replicating portfolio at time $n = 1$. (Be sure you show the method/process you used to arrive at this answer. Unsupported answers will receive no credit.)

3. Let $p \in (0, 1/2)$, and let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables with $\boldsymbol{P}(\xi_1 = 1) = \boldsymbol{P}(\xi_1 = -1) = p$ and $\boldsymbol{P}(\xi_1 = 0) = 1 - 2p$. Let $b \in \mathbb{N}$, $b > 1$, $X_0 \in (0, b) \cap \mathbb{N}$, and define $X_{n+1} = X_n + \xi_{n+1}$.

10 (a) Find $\alpha, \beta \in \mathbb{R}$ so that $\alpha b^2 + \beta b = 0$ and the process $M_n = \alpha X_n^2 + \beta X_n - n$ is a martingale.

10 (b) Let $\tau = \min\{n \in \mathbb{N} \mid X_n \notin (0, b)\}$. Find $\boldsymbol{E}\tau$.
[Note: Even though the stopping time τ is not bounded, you may assume the optional sampling theorem applies.]