21-370 Discrete Time Finance: Midterm 1.

2021-09-29

- *This is an open book test. You may use your notes, homework solutions, books, and/or online resources (including software) while doing this exam.*
- *You may* not*, however, seek or receive assistance from a live human during the exam. This includes in person assistance, instant messaging, and/or posting on online forums / discussion boards. (Searching discussion boards is OK, though.)*
- *Unsupported (or unreadable) answers will not receive credit. Show your work and explain your solution.*
- *You must record yourself (audio, video and screen) and share it with me as instructed by email.*
- *Late submissions will not be accepted. Please ensure you allow yourself ample time to scan your exam, otherwise you will get zero credit.*
- *You have 60 minutes. The exam has a total of 4 questions and 40 points.*
- *The questions are roughly in order of difficulty; though depending on your intuition, some of you may feel differently. Nevertheless, I suggest looking over the entire exam before starting. Good luck* ☺.

10 1. Consider a market consisting of a money market account with interest rate $r = 1/4$ and a stock. The stock price is initially \$12. At every time step we flip a coin which lands heads with probability $4/9$ and tails with probability $5/9$, and is independent of all previous coin flips. If the coin lands heads, the stock price increases by 50% (i.e. the new stock price is 1.5 times the old stock price). If the coin lands tails, the stock price stays the same. A European put option with strike price \$20 and maturity $N = 3$ is a security that gives the holder to the right to sell the stock at price \$20 at the maturity time. What is the arbitrage free price of this option at time $n = 1$.

Your answer may depend on the outcome of the coin tosses. You should express all quantities arising in your answer as decimal numbers, correct to two decimal places. That is, don't write $2(1.23^{-10} + 3)$ if the first coin is heads in your answer. Instead evaluate this, and get a decimal number correct to two decimal places.

10 2. You play a coin flip game which pays you \$1 if the coin lands heads, and costs you \$1 if the coin lands tails. You know that the first two coin flips are independent and fair. For the third coin toss, the rules are as follows:

(i) If the first two coin flips are both heads, then the coin is switched out for a biased coin which lands heads 40% of the time and tails 60% of the time.

(ii) If the first two coin flips are both tails, then the coin is switched out for a biased coin which lands heads 60% of the time and tails 40% of the time.

(iii) Otherwise a fair coin is used.

Let $M_0 = 0$ and M_n be your wealth after playing this game n times. (So $M_1 = 1$ if the first coin flip is heads, $M_2 = 2$ if the first two coin flips are heads, etc.) Compute $\boldsymbol{E}M_3$ and $\boldsymbol{E}_1 M_3$.

10 3. Suppose X is a $\mathcal{F}_N$-measurable random variable, $n \leqslant N$, $\alpha \in \mathbb{R}$ and let $A = \{\boldsymbol{E}_n X = \alpha\}$. If

$$\boldsymbol{P}(X = 2) = 0.3 \qquad \boldsymbol{P}(X = 3) = 0.7 \qquad \boldsymbol{P}(\{X = 2\} \cap A) = 0.2 \qquad \boldsymbol{P}(A) = 0.6\,,$$

then find α.

10 4. Let $X_1, \ldots, X_N$ be independent, identically random variables with $\boldsymbol{E}X_1 = 0$, $\boldsymbol{E}(X_1^2) = 1$ and $\boldsymbol{E}(X_1^3) = 0$. Let $M_0 = 0$, and for $n \in \{0, \ldots, N-1\}$ define $M_{n+1} = M_n + X_{n+1}$. For $\alpha, \beta \in \mathbb{R}$, define $Y_0 = 0$ and

$$Y_{n+1} = Y_n + (M_{n+1}^3 - M_n^3) + \alpha M_n + \beta n \qquad \text{for all } n \in \{0, \ldots, N-1\}\,.$$

Find all α, β so that is the process Y a martingale.