Lecture 36 (12/3). Please enable video if you can .

Thm (last time): $\xi_n \longrightarrow$ iid $\quad P(\xi_n = 1) = P(\xi_n = -1) = \frac{1}{2}$

$$S_n = \sum_{1}^{n} \xi_k \qquad S_0 = 0$$

S is rec ~~at~~ at 0 if

$$P\left(\{S_n = 0 \text{ for infinitely many } n\}\right) = 1$$

Thm (last time): In dim $d = 1$ or 2, the RW is rec.

In dim $d \geq 3$ the RW is NOT rec.

- Let $\tau_1 = \min\{n > 0 \mid S_n = 0\}$, be the first time S returns to 0.
- Let $\tau_2 = \min\{n > \tau_1 \mid S_n = 0\}$, be the first time after τ_1 that S returns to 0.
- Let $\tau_{k+1} = \min\{n > \tau_k \mid S_n = 0\}$, be the first time after τ_k that S returns to 0.

Lemma 9.5. *S is recurrent at 0 if and only if $\boldsymbol{P}(\tau_0 < \infty) = 1$.*

Pf Last time.

Lemma 9.6. $\boldsymbol{P}(\tau_0 < \infty) = 1$ *if and only if* $\sum_{n=0}^{\infty} \boldsymbol{P}(S_n = 0) = \infty$.

Proof.

(By 9.5, $\sum P(S_n = 0) = \infty \iff S$ is recurrent at 0).

Pf: (1) $E(\#\text{times } n \ni S_n = 0) = \sum_{n=0}^{\infty} P(S_n = 0)$

($\because$ $E(\#\text{times } n \ni S_n = 0) = E\left(\sum_n \mathbb{1}_{\{S_n = 0\}}\right)$

$= \sum_n E\,\mathbb{1}_{\{S_n=0\}} =$

② $E(\#\text{times } m \ni S_m = 0) = \dfrac{1}{1 - P(\tau_1 < \infty)}$

Pf: $E(\#\text{times } m \ni S_m = 0) = E\left(\sum_{n=1}^{\infty} \mathbb{1}_{\{\tau_n < \infty\}}\right)$

$= \sum_{n=1}^{\infty} P(\tau_n < \infty) = \sum_{n=1}^{\infty} P(\tau_1 < \infty)^n$

$= \dfrac{P(\tau_1 < \infty)}{1 - P(\tau_1 < \infty)}$ $\quad \left(\because P(\tau_2 < \infty) = P(\tau_1 < \infty)^2\right)$

Equate 1 & 2:

$$E(\#\text{ times } n \ni S_n = 0) = \sum P(S_n = 0) = \frac{P(\tau_1 < \infty)}{1 - P(\tau_1 < \infty)}$$

$$\Rightarrow \quad \sum P(S_n = 0) = \infty \iff P(\tau_1 < \infty) = 1$$

QED.

Theorem 9.7. $\boldsymbol{P}(S_{2m}=0) = \boxed{O(1/m^{d/2})}$. *Consequently, the random walk is* recurrent *for* $d \leqslant 2$, *and* transient *for* $d \geqslant 3$.

S_n = RW in d dim.

Assume $P(S_{2n}=0) = O\left(\frac{1}{n^{d/2}}\right)$.

Then S is rec at 0 $\iff \sum P(S_n=0) = \infty$

$\iff \sum P(S_{2n}=0) = \infty$

$\iff \sum_{n=1}^{\infty} \frac{1}{n^{d/2}} = \infty$

Note Comparison test : $\sum \frac{1}{n^s} < \infty \iff s > 1$

$\therefore$ S is rec at $0 \iff d/2 \leq 1 \iff d \leq 2$ QED.

Compute $P(S_{2n} = 0)$.

$d = 1$: $P(S_{2n} = 0) = \binom{2n}{n} \cdot \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^n$

$$= \frac{1}{2^{2n}} \binom{2n}{n} = \frac{1}{2^{2n}} \frac{(2n)!}{(n!)^2} \quad \ldots (*)$$

Lemma 9.8 (Sterling's formula). *For large n, we have*

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n = \sqrt{2\pi}\exp\left(\underbrace{n \ln n - n} + \frac{\ln n}{2}\right).$$

(Intuition: $n! = \prod_{k=1}^{n} k = e^{\sum_{k=1}^{n} (\ln k)}$

$$\sum_{k=1}^{n} \ln k \approx \int_1^n \ln x \, dx = \underbrace{n \ln n - n}$$)

Using Sterling on (*) $\Rightarrow P(S_{2n}=0) \approx \frac{1}{2^{2n}} \cdot \left(\frac{\sqrt{2\pi\, 2n}\left(\frac{2n}{e}\right)^{2n}}{\left[\sqrt{2\pi\, n}\left(\frac{n}{e}\right)^{n}\right]^2} \right).$

~~*Proof of Theorem 9.7 for d = 1:*~~

$$\approx \frac{1}{2^{2n}} \frac{\sqrt{2}}{\sqrt{2\pi}} \frac{\sqrt{n}}{n} \frac{(2n)^{2n}}{n^{2n}}$$

$$\approx \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{n}}$$ as claimed

$\Rightarrow$ Thm for $d = 1$

QED

Remark 9.9. Recall the Gambler's ruin example (Question 6.50): Let ξ_n be i.i.d. random variables with mean 0, and let $X_n = \sum_1^n \xi_k$. Let $\tau = \min\{n \mid X_n = 1\}$. Theorem 9.7 proves $\tau < \infty$ almost surely. We proved earlier $\boldsymbol{E} X_\tau = 1$ and $\lim_{N\to\infty} \boldsymbol{E} X_{\tau\wedge N} = 0$.

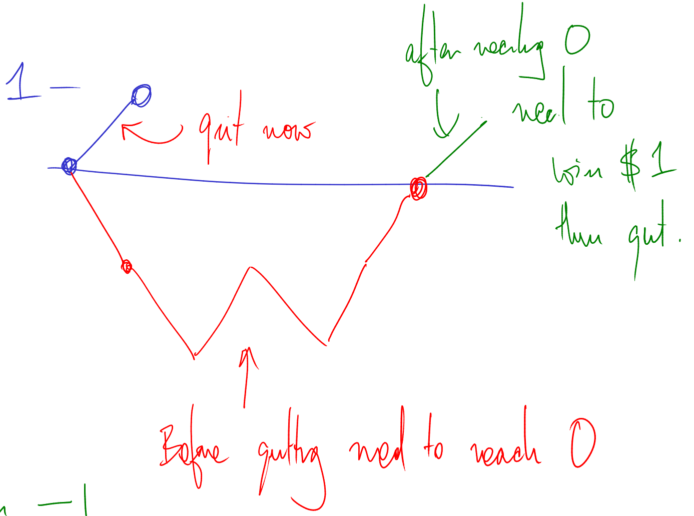
Theorem 9.10. *Consider the Gamblers ruin example, with* $\tau = \min\{n \mid X_n = 1\}$. *Then*

$$\boxed{\boldsymbol{E}\tau = \infty} \quad \text{and} \quad \boldsymbol{P}(\tau = 2n - 1) = (-1)^{n-1}\binom{1/2}{n} \approx \frac{C}{n^{3/2}}$$

Remark 9.11. Let $M_n = \min\{X_{\tau \wedge k} \mid k \leqslant n\}$. Then $\boldsymbol{E}M_\tau = -\infty$. Thus, this strategy will take (on average) an infinite time before you win \$1. During that time your expected maximum loss is $-\infty$.

Lemma 9.12. *Let* $F(x) = \boldsymbol{E}x^\tau$. *Then* $\boxed{F(x) = \frac{1}{x}(1 - \sqrt{1 - x^2})}$.

Pf:



$$F(x) = E x^\tau$$

$$= \frac{1}{2}x^1 + \frac{1}{2}E x^{\tau' + \tau'' + 1}$$

τ' = first time X_n reaches 0 starting from -1

τ'' = first time after τ', X_n reaches $+1$

dist of τ' = dist of τ'' = dist of τ & τ'' & τ' are ind

$$\Rightarrow E x^{\tau} = \frac{1}{2} x + \frac{1}{2} \quad x \, E x^{\tau'} \, E x^{\tau''}$$

$$= \frac{1}{2} x + \frac{x}{2} \left(E x^{\tau} \right)^2 \Rightarrow F(x) = E x^{\tau} = \frac{x}{2} + \frac{x}{2} F(x)^2$$

Solve for $F(x)$ & get the formula.

$F(x) =$

Proof of Theorem 9.10

① $E\tau = \infty$.

Pf: $F(x) = E x^{\tau} = \dfrac{1 - \sqrt{1-x^2}}{x}$

diff wrt x: $F'(x) = E \tau x^{\tau - 1}$

Put $x=1$: $F'(1) = E\tau$

Find $F'(1)$ from formula & get $E\tau = +\infty$!

② Find $P(\tau = n)$

Know $E\, x^{\tau} = F(x) = \dfrac{1 - \sqrt{1 - x^2}}{x}$

$$E\, x^{\tau} = 1\, P(\tau=0) + x\, P(\tau=1) + x^2\, P(\tau=2) + \cdots$$

$$= \sum_{k=0}^{\infty} x^k \underbrace{P(\tau = k)}$$

Also $F(x) = E\, x^{\tau} = \dfrac{1 - \sqrt{1 - x^2}}{x}$. Taylor series & find a formula.