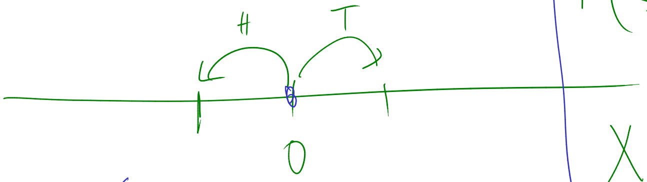
Lecture 35 (12/1). Please enable video if you can.

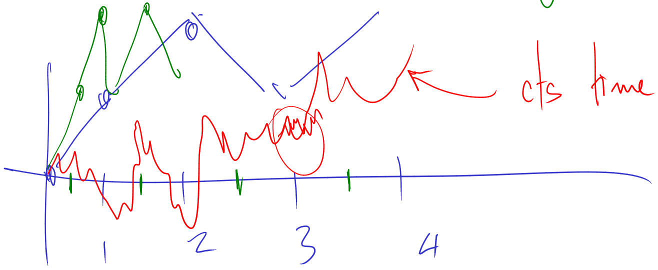
"Random walk in continuous time"



$\xi_n \longrightarrow$ iid

$$P(\xi_n = 1) = P(\xi_n = -1) = \frac{1}{2}.$$

$$X_n = \sum_{k=1}^{n} \xi_k$$



cts time limit

Cts time RW: flip a coin every $\frac{1}{N}$ seconds & take a step of size $\frac{1}{\sqrt{N}}$ left or right

Merton

8. Black-Scholes Formula

(1) Suppose now we can trade *continuously in time.*

(2) Consider a market with a bank and a stock, whose spot price at time t is denoted by S_t.

(3) The *continuously compounded interest rate* is r (i.e. money in the bank grows like $\partial_t C(t) = rC(t)$).

$$C_t = C_0 \cdot e^{rt}$$

(4) Assume liquidity, neglect transaction costs (frictionless), and the borrowing/lending rates are the same.

(5) In the Black-Scholes setting, we model the stock prices by a *Geometric Brownian motion* with parameters α (the mean return rate) and σ (the volatility).

(6) The price at time t of a European call with maturity T and strike K is given by

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)),$$

where
$$d_\pm = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right), \qquad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy.$$

$\tau = T - t$

CDF of normal dist

(7) Can be obtained as the limit of the Binomial model as $N \to \infty$ by choosing:

$$r_{\text{binom}} = \frac{r}{N}, \qquad u = u_N = 1 + \frac{r}{N} + \frac{\sigma}{\sqrt{N}} \qquad d = d_N = 1 + \frac{r}{N} - \frac{\sigma}{\sqrt{N}}$$

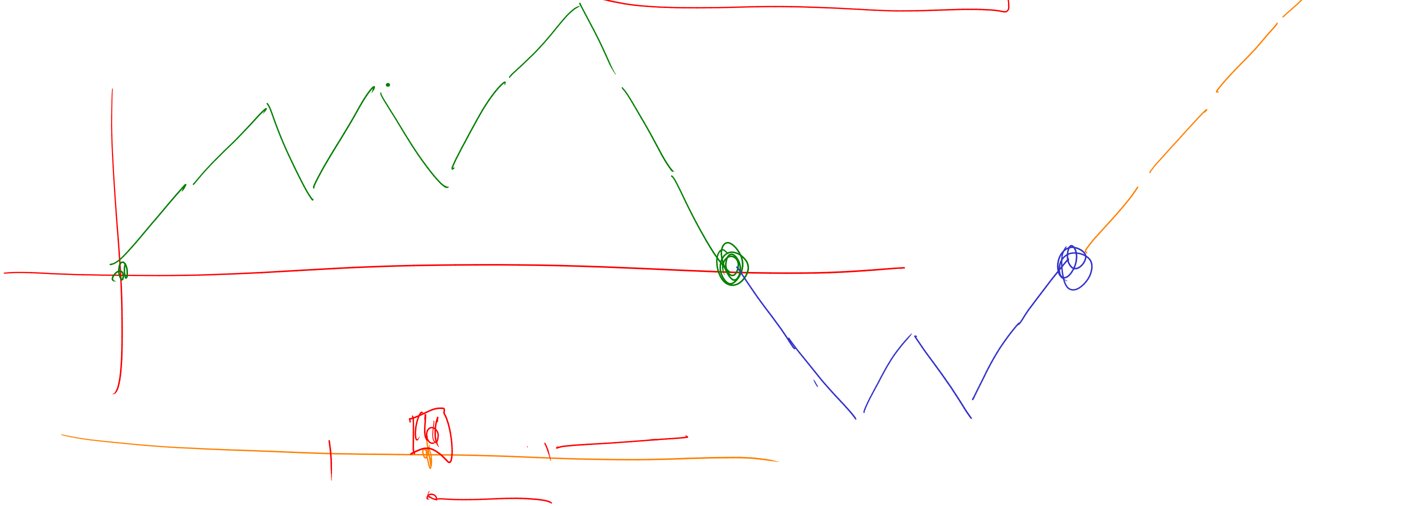
$N \to \infty$!

$$u_N - d_N = \frac{2\sigma}{\sqrt{N}}$$

9. Recurrence of Random Walks

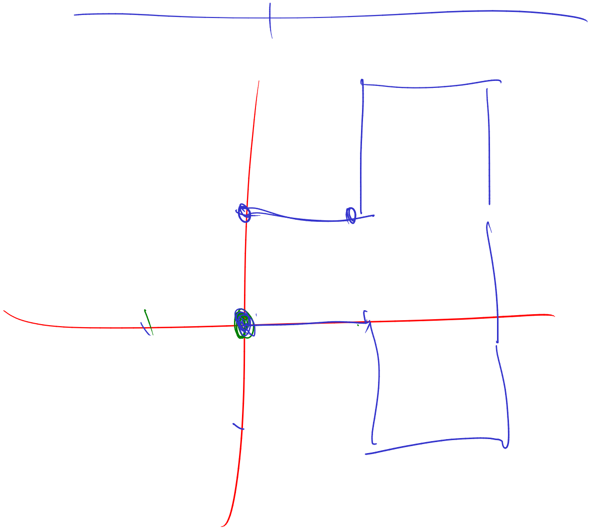
- Let ξ_n be a sequence of i.i.d. coin flips with $\boldsymbol{P}(\xi_n = 1) = \boldsymbol{P}(\xi_n = -1) = 1/2$.
- Simple random walk: $S_n = \sum_1^n \xi_k$ (i.e. $S_0 = 0$, $S_{n+1} = S_n + \xi_{n+1}$).

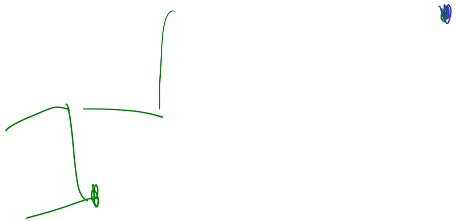
Definition 9.1. The process S_n is recurrent at 0 if $\boxed{\boldsymbol{P}(S_n = 0 \text{ infinitely often }) = 1}$



Question 9.2. *Is the random walk (in one dimension) recurrent at* 0*? How about at any other value?*

Question 9.3. *Say* ξ_n *are i.i.d. random vectors in* $\mathbb{R}^d$ *with* $\boldsymbol{P}(\xi_n = \pm e_i) = \frac{1}{2d}$*. Set* $S_n = \sum_1^n \xi_k$*. Is* S_n *recurrent at* 0*?*





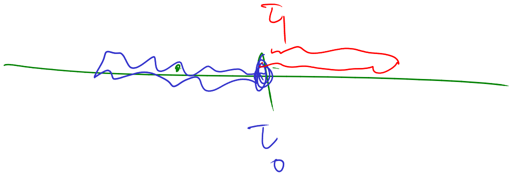
Theorem 9.4. *The simple random walk in* $\mathbb{R}^d$ *is recurrent for* $d = 1, 2$ *and transient for* $d \geqslant 3$.

Proof blow.

- Let $\tau_0 = \min\{n \mid S_n = 0\}$, be the first time S returns to 0.
- Let $\tau_1 = \min\{n \geqslant \tau_0 \mid S_n = 0\}$, be the first time after τ_0 that S returns to 0.
- Let $\tau_{k+1} = \min\{n \geqslant \tau_k \mid S_n = 0\}$, be the first time after τ_k that S returns to 0.

Lemma 9.5. *S is recurrent at 0 if and only if* $\boldsymbol{P}(\tau_0 < \infty) = 1$.

$P(S_n \text{ returns to } 0 \text{ infinitely often}) =$



Pf: (1) Say S is rec at 0

$\Rightarrow P(S_n \text{ returns to } 0 \text{ i.o.}) = 1 \Rightarrow P(S_n \text{ returns to } 0 \text{ once}) = 1$

$\Rightarrow P(\tau_0 < \infty) = 1$

(2) Conversely say $P(\tau_0 < \infty) = 1$.

Note $P(\tau_1 < \infty) = P(\tau_0 < \infty)^2$

Odds of returning twice

$$= \underbrace{\text{Odds of returning once}}_{P(\tau_0 < \infty)} \;\&\; \boxed{\underbrace{\text{then returning a second time}}_{P(\tau_0 < \infty)}}$$

Ind

By ind: $P(\tau_n < \infty) = P(\tau_0 < \infty)^n = 1 \Rightarrow P(S_n \text{ returns to } 0 \text{ i.o.}) = 1.$

Lemma 9.6. $\boxed{\boldsymbol{P}(\tau_0 < \infty) = 1}$ *if and only if* $\boxed{\sum \boldsymbol{P}(S_n = 0) = \infty}$.

Proof.

i.e. $\boxed{\sum_{n=0}^{\infty} P(\cancel{P} S_n = 0) = \infty}$ $\iff P(\tau_0 < \infty) = 1$

$\overset{\text{Lemma}}{\iff} P(S_n \text{ returns to } 0 \text{ i.o.}) = 1$

(i.e. S_n is recurrent at 0).