Lecture 34 (11/29). Please enable video if you can.

Recall: FTAP 1: No arb $\iff$ Existence of a RNM.

FTAP 2: No arb & completeness $\iff$ Existence & uniq of a RNM

Every sec can be replicated.

S^0 ← Bank

S^1

$\vdots$

S^d

Stocks.

$$D_n = \frac{1}{S^0_n}$$

7.4. Examples and Consequences.

Proposition 7.16. *Suppose the market model Section 7.1 is complete and arbitrage free, and let $\tilde{\boldsymbol{P}}$ be the unique risk neutral measure. If $D_n X_n$ is a $\tilde{\boldsymbol{P}}$ martingale, then X_n must be the wealth of a self financing portfolio.*

Remark 7.17. We've already seen in Lemma 7.5 that if a (not necessarily unique) risk neutral measure exists, then the discounted wealth of any self financing portfolio must be a martingale under it.

Remark 7.18. All pricing results/formulae we derived for the Binomial model that only relied on the analog of Proposition 7.16 will hold in complete arbitrage free markets.

Pf of Prop 7.16. Know $D_n X$ is a $\tilde{P}$ mg

NTS: X_n = wealth of a self fin port

i.e. NTF a trading strat Δ_n + $X_n = \Delta_n \cdot S_n$

self fin cond. & $\boxed{\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}}$

$$\Delta_n = (\Delta_n^0, \Delta_n^1 \dots \Delta_n^d), \quad \Delta_n \cdot S_n = \sum_{i=0}^{d} \Delta_n^i S_n^i = \text{total wealth}$$

Market is complete : $\Rightarrow \exists \underset{\sim}{\Delta_{N-1}} \; (\delta_{N-1} - \text{mess}) \ni \underline{\underline{X_N}} = \Delta_{N-1} \cdot \underline{\underline{S_N}}$

Claim $X_{N-1} = \Delta_{N-1} \cdot S_{N-1}$

Pf: know $X_N = \Delta_{N-1} \cdot S_N$

$$\Rightarrow \quad D_N X_N = \Delta_{N-1} \cdot (D_N S_N)$$

$$\Rightarrow \quad \tilde{E}_{N-1}(D_N X_N) = \Delta_{N-1} \cdot \left(\tilde{E}_{N-1}(D_N S_N)\right)$$

$$\Rightarrow \quad D_{N-1} X_{N-1} = \Delta_{N-1} \cdot (D_{N-1} S_{N-1}) \qquad \left(\because D_n X_n \text{ is a } \tilde{P} \text{ mg} \right.$$

$$\because \text{By def of } \tilde{P}$$

$$\Rightarrow \quad X_{N-1} = \Delta_{N-1} \cdot S_{N-1} \quad \cdots \; (*) \qquad \left. D_n S_n \text{ is a } \tilde{P} \text{ mg} \right)$$

QED

Recall: Def of completeness $\Rightarrow \exists\, \Delta_{N-2}$ + $X_{N-1} = \Delta_{N-2} \cdot S_{N-1}$ $\cdots (**)$

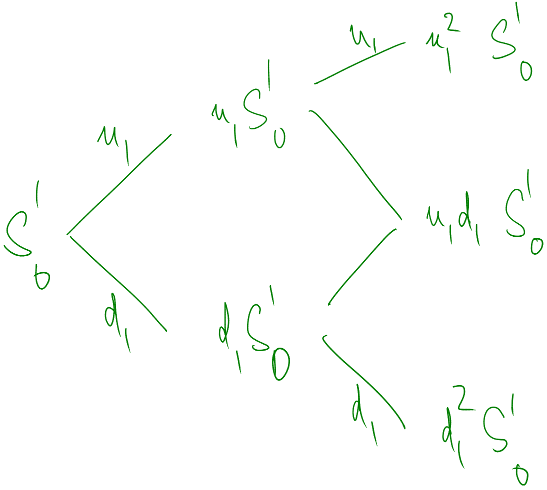
By above, get $X_{N-2} = \Delta_{N-2} \cdot S_{N-2}$

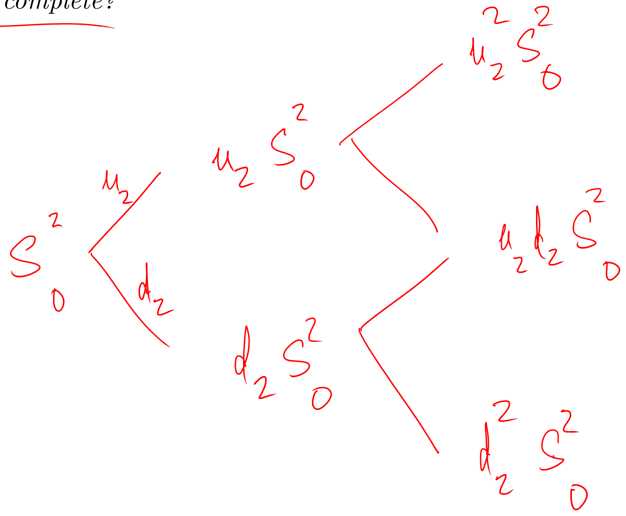
Keep going & get the trading strat (Δ_n)

Note: this trading strat is self fin

by equating $(*)$ & $(**)$ (& repeating)

Question 7.19. *Consider a market consisting of a bank with interest rate r, and two stocks with price processes S^1, S^2. At each time step we flip two independent coins. The price of the i-th stock ($i \in \{1, 2\}$) changes by factor u_i, or d_i depending on whether the i-th coin is heads or tails. When is this market arbitrage free? When is this market complete?*





Arb free $\iff$ $\exists$ a RNM.

$\tilde{p}, \tilde{q}$ RN Prob of heads & tails.

Need ① $\tilde{p} + \tilde{q} = 1$, $\tilde{p}, \tilde{q} > 0$

② $\tilde{E}_n S^1_{n+1} = (1+r) S^1_n \longrightarrow \tilde{E}_n S^1_{n+1} = \tilde{p}\, u_1 S^1_n + \tilde{q}\, d_1 S^1_n$

$= (\tilde{p}\, u_1 + \tilde{q}\, d_1) S^1_n$

$\overset{\text{Want}}{=} (1+r) S^1_n$

③ $\tilde{E}_n S^2_{n+1} = (1+r) S^2_n$

$\Rightarrow$ Shall have

$$\textcircled{1} \quad \tilde{p} + \tilde{q} = 1$$

$$\left.\begin{aligned} &\textcircled{2} \quad u_1 \hat{p} + d_1 \hat{q} = 1 + r \\ &\textcircled{3} \quad u_2 \hat{p} + d_2 \hat{q} = 1 + r \end{aligned}\right\}$$

$$\boxed{\text{In matrix form: } \begin{pmatrix} 1 & 1 \\ u_1 & d_1 \\ u_2 & d_2 \end{pmatrix} \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} 1 \\ 1+r \\ 1+r \end{pmatrix}} \quad \cdots \textcircled{\ast}$$

Have a RNM $\Leftrightarrow$ we have > 0 ~~sol~~ solutions to $(*)$.

Linear Alg: 3 eq & 2 unknowns $\longrightarrow$ may not have a soln.

Will only have solutions if the system is consistent

e.g if the 3^{rd} eq is a comb of the first 2.

Finally: Need S^2 to be a combination of Bank & S^1

Rule of thumb: To get a complete & arb free market with d stocks, M.M., will need to roll a $d+1$ sided die

(Write down eq for RNM.

get (# stocks + 1) eqns in (# faces of the die) unknowns)

Question 7.20. *Consider now repeated rolls of a 3-sided die and for $i \in \{1,2\}$, suppose $S^i_{n+1} = f_{i,j} S^i_n$, if $\omega_{n+1} = j$. How do you find the risk neutral measure? Find conditions when this market is complete and arbitrage free.*

$(j \in \{1, 2, 3\})$.

RNM: $\tilde{p}_i$ = Prob the die rolls i

System of eq:

(1) $\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 = 1$

(2) $\tilde{p}_1 f_{11} + \tilde{p}_2 f_{12} + \tilde{p}_3 f_{13} = 1+r$

(3) $\tilde{p}_1 f_{21} + \tilde{p}_2 f_{22} + \tilde{p}_3 f_{23} = 1+r$

$$\Longleftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \end{pmatrix} \underbrace{\begin{pmatrix} \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \end{pmatrix}} = \begin{pmatrix} 1 \\ 1+r \\ 1+r \end{pmatrix}$$

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* Finding Δ:

Knows $\quad V_0 = \frac{1}{1+r} \tilde{E}_1 V_1 \quad \longrightarrow$ find V_1

Q: $\Delta_0 = ?$

Known

Choose Δ_0 so that $\Delta_0 \cdot S_1 = V_1$

$$\Delta_0^0 S_1^0(1) + \Delta_0^1 S_1^1(1) + \Delta_0^1 S_1^1(1) = V_1(1)$$

$$\sum_{i=0}^{d} \Delta_0^i S_1^i(j) = V_1(j) \longleftarrow$$ $d+1$ equations (one for each j)

$$\begin{pmatrix} 1+r & f_{1,1} & f_{21} \\ 1+r & f_{1,2} & f_{22} \\ 1+r & f_{1,3} & f_{23} \end{pmatrix} \begin{pmatrix} \Delta_0^0 \\ \Delta_0^1 \\ \Delta_0^2 \end{pmatrix} = \begin{pmatrix} V_1(1) \\ V_1(2) \\ V_1(3) \end{pmatrix}$$

$d+1$ unknowns (each Δ_0^i)

8. Black-Scholes Formula

(1) Suppose now we can trade *continuously in time.*
(2) Consider a market with a bank and a stock, whose spot price at time t is denoted by S_t.
(3) The *continuously compounded interest rate* is r (i.e. money in the bank grows like $\partial_t C(t) = rC(t)$).
(4) Assume liquidity, neglect transaction costs (frictionless), and the borrowing/lending rates are the same.
(5) In the *Black-Scholes* setting, we model the stock prices by a *Geometric Brownian motion* with parameters α (the mean return rate) and σ (the volatility).
(6) The price at time t of a European call with maturity T and strike K is given by

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)),$$

$$\text{where} \quad d_\pm = \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big), \qquad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy.$$

(7) Can be obtained as the limit of the Binomial model as $N \to \infty$ by choosing:

$$r_{\text{binom}} = \frac{r}{N}, \qquad u = u_N = 1 + \frac{r}{N} + \frac{\sigma}{\sqrt{N}} \qquad d = d_N = 1 + \frac{r}{N} - \frac{\sigma}{\sqrt{N}}$$

9. **Recurrence of Random Walks**

- Let ξ_n be a sequence of i.i.d. coin flips with $\boldsymbol{P}(\xi_n = 1) = \boldsymbol{P}(\xi_n = -1) = 1/2$.
- Simple random walk: $S_n = \sum_1^n \xi_k$ (i.e. $S_0 = 0$, $S_{n+1} = S_n + \xi_{n+1}$).

Definition 9.1. The process S_n is recurrent at 0 if $\boldsymbol{P}(S_n = 0 \text{ infinitely often })$.

Question 9.2. *Is the random walk (in one dimension) recurrent at* 0*? How about at any other value?*

Question 9.3. *Say* ξ_n *are i.i.d. random vectors in* $\mathbb{R}^d$ *with* $\boldsymbol{P}(\xi_n = \pm e_i) = \frac{1}{2d}$. *Set* $S_n = \sum_1^n \xi_k$. *Is* S_n *recurrent at* 0*?*

Theorem 9.4. *The simple random walk in* $\mathbb{R}^d$ *is recurrent for* $d = 1, 2$ *and transient for* $d \geqslant 3$.

- Let $\tau_0 = \min\{n \mid S_n = 0\}$, be the first time S returns to 0.
- Let $\tau_1 = \min\{n \geqslant \tau_0 \mid S_n = 0\}$, be the first time after τ_0 that S returns to 0.
- Let $\tau_{k+1} = \min\{n \geqslant \tau_k \mid S_n = 0\}$, be the first time after τ_k that S returns to 0.

Lemma 9.5. *S is recurrent at 0 if and only if $\boldsymbol{P}(\tau_0 < \infty) = 1$.*

Lemma 9.6. *$\boldsymbol{P}(\tau_0 < \infty) = 1$ if and only if $\sum \boldsymbol{P}(S_n = 0) = \infty$.*

Proof.

Theorem 9.7. $\boldsymbol{P}(S_{2m} = 0) = O(1/m^{d/2})$. *Consequently, the random walk is* recurrent *for* $d \leqslant 2$, *and* transient *for* $d \geqslant 3$.

Lemma 9.8 (Sterling's formula)**.** *For large n, we have*

$$n! \approx \sqrt{2\pi} \exp\left(n \ln n - n + \frac{1}{2}\right).$$

Proof of Theorem 9.7 for $d = 1$: