Lecture 33 (11/22). Please enable video if you can.

So far: Binomial model $0 < d < 1+r < u \longrightarrow$ "complete & arb free"

Every security can be replicated.

FTAP 1: Arb free $\Longleftrightarrow \exists$ a RNM.

7.3. Second fundamental theorem.

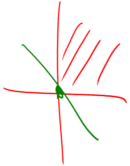
Definition 7.13. A market is said to be *complete* if every derivative security can be hedged.

Theorem 7.14. *The market defined in Section 7.1 is complete and arbitrage free if and only if there exists a* unique *risk neutral measure.*

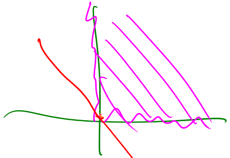
1 bank, d stocks $(S^0, S^1 \dots S^d)$

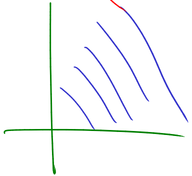
i.e. If V_N is the payoff of any security then $\exists$ a self fin port (X_n, Δ_n) such that $X_N = V_N$.

Wealth, trading strat



$(\Rightarrow \forall n \le N$, X_n = unique AFP of the security$)$

Recall: $\mathbb{R}^M$: $\overline{Q}$ =  $= \{v \in \mathbb{R}^M \mid v_i \geq 0 \ \forall i\}$

$\mathring{Q}$ =  $= \{v \in \mathbb{R}^M \mid v_i > 0 \ \forall i\}$

Lemma: $V \subseteq \mathbb{R}^M$ a subspace.

① $V \cap \overline{Q} = \{0\} \iff \exists\, \hat{u} \in \mathring{Q}$ s.t. $\hat{u} \perp V$ & $|\hat{u}| = 1$

Use today → ② Say $V \cap \overline{Q} = \{0\}$. Then $\exists$ a unique $\hat{u} \in \mathring{Q}$ s.t. $\hat{u} \perp V$, $|\hat{u}| = 1 \iff \dim(V) = M-1$

Lemma 7.15. *The market is complete if and only if for every $\mathcal{F}_{n+1}$-measurable random variable X_{n+1}, there exists a (not necessarily unique) $\mathcal{F}_n$ measurable random vector $\Delta_n = (\Delta_n^0, \ldots, \Delta_n^d)$ such that $X_{n+1} = \boxed{\Delta_n} \cdot S_{n+1}$.*

Pf: (1) Say $n+1 = N$

Say market is complete.

X_N is some $\mathcal{F}_N$-meas RV.

Let $V_N = X_N$ = payoff of a security at time N

Complete $\Rightarrow$ can be replicated.

$\Rightarrow \exists$ a self fin port (X_n, Δ_n) + $X_N = V_N$.

a

Know ① Wealth at time n = $X_n = \Delta_n \cdot S_n = \sum_{i=0}^{d} \Delta_n^i S_n^i$

& ② $\Delta_n \cdot S_{n+1} = \underbrace{\Delta_{n+1} \cdot S_{n+1}}_{X_{n+1}}$

$\Rightarrow X_{n+1} = \Delta_n \cdot S_{n+1}$

$\Rightarrow V_N = X_N = \underline{\underline{\Delta_{N-1}}} \cdot S_N .$

Conversely: Say $\forall$ $\mathcal{F}_{n+1}$ meas RV V_{n+1}, $\exists$ Δ_n ($\mathcal{F}_n$-meas) $\ni$ $V_{n+1} = \Delta_n \cdot S_{n+1}$

NTS: Market is complete.

Let V_N = payoff of some security

NTF (X_n, Δ_n) self fin $\ni$ $X_N = V_N$.

Pf: By assumption $\exists$ Δ_{N-1} $\ni$ $V_N = \Delta_{N-1} \cdot S_N$

Let $X_N = V_N$. Let $\underline{X_{N-1}} = \Delta_{N-1} \cdot S_{N-1}$

Assumption $\Rightarrow \exists \Delta_{N-2}$ ($\mathcal{F}_{N-2}$-meas) $\ni X_{N-1} = \Delta_{N-2} \cdot S_{N-1}$

Set $X_{N-2} = \Delta_{N-2} \cdot S_{N-2}$ & continue.

Self fin:

$$\Delta_{N-2} \cdot S_{N-1} \;\underline{\underline{\text{Want}}}\; \Delta_{N-1} \cdot S_{N-1}$$

Note: $X_{N-1} = \Delta_{N-2} \cdot S_{N-1}$ & by choice $X_{N-1} = \Delta_{N-1} \cdot S_{N-1}$

$$\Rightarrow \Delta_{N-2} \circ S_{N-1} = \Delta_{N-1} \circ S_{N-1}$$

(repeat $\Rightarrow$ self fin)

QED.

Proof of Theorem 7.14 NTS RNM unique $\iff$ complete & arb free.

Case 1: $N = 1$.

Let $V = \{ \Delta_0 \cdot S_1 \mid \Delta_0 \cdot S_0 = 0 \}$

$$= \left\{ \begin{pmatrix} \Delta_0 \cdot S_1(1) \\ \Delta_0 \cdot S_1(2) \\ \vdots \\ \Delta_0 \cdot S_1(M) \end{pmatrix} \in \mathbb{R}^M \;\middle|\; \Delta_0 \cdot S_0 = 0 \right\} \subseteq \mathbb{R}^M \text{ (subspace)}.$$

no restriction

Let $U = \{ \Delta_0 \cdot S_1 \mid \Delta_0 \cdot S_0 \in \mathbb{R} \} \subseteq \mathbb{R}^M$ (subspace).

Recall: If $\exists\, \hat{u} \in \mathring{Q}$ (i.e. $\underline{\hat{u}_i > 0}$) $\ni$ $\hat{u} \perp V$

then can use $\hat{u}$ to make a RNM.

(Chose $\tilde{p}_1(i) \underline{\underline{\text{last time}}} \dfrac{\hat{u}_i}{\sum_{j=1}^{M} \hat{u}_j}$ got a RNM

& conversely $\tilde{p}_1(i)$ is the RN probability that 1^{st} die rolls i

then $\begin{pmatrix} \tilde{p}_1(1) \\ \vdots \\ \tilde{p}_1(M) \end{pmatrix} \in \mathring{Q}$ & is $\perp V$)

Say Market is complete & arb free.

$\Rightarrow \quad V \cap \bar{Q} = \{0\}$ (follows since no arb)

& $\quad \mathcal{U} = \mathbb{R}^M$ (follows since the market is complete & Lemma 7.11).

In this case

$$\dim(V) = \dim(\mathcal{U}) - 1 \quad \text{(on HW)}$$

$\Rightarrow \quad \dim(V) = M - 1 \quad$ (& $V \cap \bar{Q} = \{0\}$) $\Rightarrow \hat{\eta} \in \mathring{Q}$ is unique

rescale coordinates $\Rightarrow \hat{p}_1$ is unique (i.e. RNM is unique).

Conversely: Suppose the RNM is unique.

Know if $\hat{u} \in \mathring{Q}$ & $\hat{u} \perp V$

then can rescale coordinates of $\hat{u}$ & get a RNM.

RNM unique $\Leftrightarrow$ $\hat{u}$ is unique $\Leftrightarrow$ $\dim(V) = M-1$

$\Rightarrow \dim(U) = M \overset{\text{Lemma}}{\Longrightarrow}$ Completeness!