Lecture 32 (11/19). Please enable your video if you can

FTAP 1: No arb $\Longleftrightarrow$ $\exists$ a RNM.

Simpler $\longrightarrow$ $\exists$ RNM $\Rightarrow$ No arb (done a few lectures ago)

Harder $\longrightarrow$ No arb $\Rightarrow$ $\exists$ RNM.

Last time: (1) Say $\tilde{P}$ is a measure + Whenever $\underbrace{\Delta_n \cdot S_n}_{\text{Wealth at time } n} = 0$

we have $\tilde{E}_n(\underline{\Delta_n \cdot S_{n+1}}) = 0$

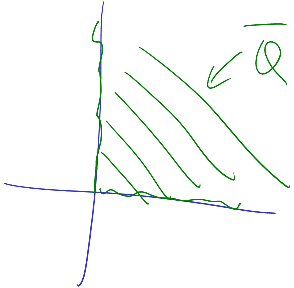
Then $\tilde{P}$ is a RNM.

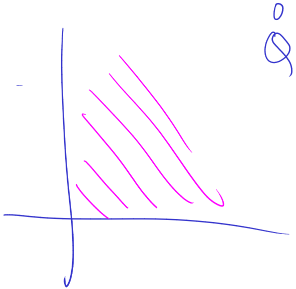
Lemma 7.11. *Define* $\bar{Q} \overset{\text{def}}{=} \{v \in \mathbb{R}^M \mid v_i \geqslant 0\ \forall i \in \{1, \ldots, M\}\}$, *and* $\mathring{Q} \overset{\text{def}}{=} \{v \in \mathbb{R}^M \mid v_i > 0\ \forall i \in \{1, \ldots, M\}\}$. *Let* $V \subseteq R^M$ *be a subspace.*

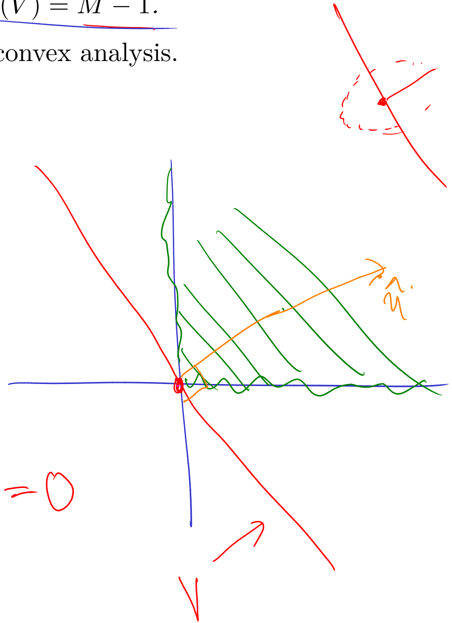
(1) $V \cap \bar{Q} = \{0\}$ *if and only if there exists* $\hat{n} \in \mathring{Q}$ *such that* $|\hat{n}| = 1$ *and* $\hat{n} \perp V$.

(2) *The unit normal vector* $\hat{n} \in \mathring{Q}$ *is unique if and only if* $V \cap \bar{Q} = \{0\}$ *and* $\dim(V) = M - 1$.

Remark 7.12. This can be proved using the *Hyperplane separation theorem* used in convex analysis.







$\bar{Q}$

Recall: $\hat{n} \perp V$ means $\forall v \in V$, $\hat{n} \cdot v = 0$

$|\hat{n}| = \left(\sum \hat{n}_i^2\right)^{1/2}$.

Proof of Theorem 7.4 (No arbitrage implies existence of a risk neutral measure).

Assume: No arb. NTS: $\exists$ a RNM.

Case I:

N = 1: Start with $X_0 = 0 = \Delta_0 \cdot S_0 \quad (\Delta_0 = (\Delta_0^0, \dots \Delta_0^d) \in \mathbb{R}^{d+1})$

Let $V = \{\Delta_0 \cdot S_1 \mid \Delta_0 \cdot S_0 = 0\} \subseteq \mathbb{R}^M$

i.e. $V = \left\{ \begin{pmatrix} \Delta_0 \cdot S_1(1) \\ \Delta_0 \cdot S_1(2) \\ \vdots \\ \Delta_0 \cdot S_1(M) \end{pmatrix} \middle| \; \Delta_0 \cdot S_0 = 0 \right\}$

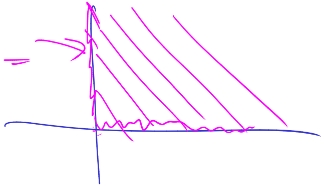
$\Delta_0 \cdot S_0 = \sum_{i=0}^{d} \Delta_0^i S_0^i$

Recall: d stocks $S^1 \dots S^d$
Bank S^0
$\}$ d+1 assets

Price changes by the roll of an M-sided die.

(i.e. $\Delta_0 \cdot S_1$ as a vector with i^{th} coordinate representing the wealth if the first die rolls i.)

Note: ① $V \subseteq \mathbb{R}^M$ is a subspace (You check it, quick).

② $V \cap \bar{Q} = \{0\}$ ($\because$ No out of $\bar{Q}$ = )

Hence: Lemma 7.11 $\Rightarrow \exists \, \hat{n} \in \mathring{Q}$ s.t. $\hat{n}(i) > 0$

($\hat{n}(i) = i^{th}$ coordinate of $\hat{n}$).

Let $\tilde{p}_1(i) = \dfrac{\hat{u}(i)}{\sum_{j=1}^{M} \hat{u}(j)}$ Claim $\tilde{p}_1(i)$ = RNP of 1^{st} die roll $= i$

(need denom to ensure $\sum_{i=1}^{M} \tilde{p}(i) = 1$)

Compute $\tilde{E}(\Delta_0 \cdot S_1)$ for any Δ_0 such that $\underbrace{\Delta_0 \cdot S_0 = 0}$

Note $\tilde{E}(\Delta_0 S_1) = \sum \tilde{p}(i) \cdot \Delta_0 \cdot S_1(i) = \sum \dfrac{\hat{u}(i)}{\left(\sum \hat{u}(j)\right)} \Delta_0 \cdot S_1(i)$

$$= \frac{1}{\left(\sum_{j} \hat{u}(j)\right)} \underbrace{\sum_{i} \hat{u}(i)\, \Delta_0 S_1(i)}_{\hat{u}\,\cdot\,(\Delta_0 S_1)}$$

$$= 0 \quad (\because \hat{u} \text{ is a normal vector})$$

By lemma from last time $\Rightarrow \hat{P}$ is a RNM. QED $(N=1)$

Case 2: $N = 2$.

Suppose $\underline{\omega_1 = 1}$ (1^{st} die already rolled 1)

Start with $\Delta_1 \in \mathbb{R}^{d+1}$ + $\Delta_1 \cdot S_1(1) = 0$

(i.e. wealth at time 1 if 1^{st} die is 1 $= 0$)

Let $V = \{ \Delta_1 \cdot S_2(1, \cdot) \mid \Delta_1 \cdot S_1(1) = 0 \}$

i.e. $V = \left\{ \begin{pmatrix} \Delta_1 \cdot S_2(1,1) \\ \Delta_1 \cdot S_2(1,2) \\ \vdots \\ \Delta_1 \cdot S_2(1,M) \end{pmatrix} \;\middle|\; \underline{\Delta_1 \cdot S_1(1) = 0} \right\}$

① $V \subseteq \mathbb{R}^M$ is a subspace

② No arb $\Rightarrow V \cap \bar{Q} = \{0\}$.

Lemma $\Rightarrow \exists\, \hat{u} \in \mathring{Q}$

Let $\tilde{p}_2(1, i) = \dfrac{\hat{u}(i)}{\sum_j \hat{u}(j)}$

③ Say $\omega_1 = 1$ & $\Delta_1 \cdot S_1(1) = 0$

Compute $\tilde{E}_1(\Delta_1 \cdot S_2)(1) = \sum \tilde{p}_2(1,i)\ \Delta_1 S_2(1,i)$

$$= \frac{1}{\sum \hat{u}(j)} \left(\hat{u} \cdot \begin{pmatrix} \Delta_1 \cdot S_2(1,1) \\ \vdots \\ \Delta_1 \cdot S_2(1,M) \end{pmatrix} \right)$$

$$= 0 \qquad (\because \hat{u} \perp V).$$

Last time's lemma $\Rightarrow \tilde{P}$ is a RNM.

[Note: $\tilde{p}(\omega) = \tilde{p}_1(\omega_1)\ \tilde{p}_2(\omega_1, \omega_2) \cdots$].