Lecture 31 (11/17). Please enable your video if you can.

Last time: Multiple assets $S^0, S^1, \dots S^d$

$\underbrace{S^1, \dots S^d}_{\text{Stocks}}$

$S^0 \to$ Bank

$S^0_n \longrightarrow$ price of 1 share of M.M acct at time n.

int rate r_n: $S^0_{n+1} = (1+r_n) S^0_n$

Discount factor $D_n = \frac{1}{S^0_n}$

Notation : sub script $\longrightarrow$ time (n)

super script i $\longrightarrow$ i^{th} stock.

<u>RNM:</u> $\tilde{P}$ $\ni$ $\forall i \in \{1, .. d\}$, $\tilde{E}_n(D_{n+1} S^i_{n+1}) = D_n S^i_n$

(Note for $i=0$, $D_n S^0_n = D_{n+1} S^0_{n+1} = 1$

$\Rightarrow \tilde{E}_n(D_{n+1} S^0_{n+1}) = D_n S^0_n$)

<u>Last time:</u> FTAP 1 : (a) If a RNM exists then there is no arb.

(proved last time)

(b) No arb $\Longrightarrow$ $\exists$ a RNM

(need not be unique)

(IOU Proof $\to$ today).

Corollary 7.6. *Suppose the market has a risk neutral measure* $\tilde{\boldsymbol{P}}$*. Let* V_N *be a* $\mathcal{F}_N$*-measurable random variable and consider an security that pays* V_N *at time* N*. Then* $V_n = D_n^{-1}\tilde{\boldsymbol{E}}_n(D_N V_N)$ *is a arbitrage free price at time* $n \leqslant N$*. (i.e. allowing you to trade this security in the market with price* V_n *at time* n *keeps the market arbitrage free).*

Remark 7.7. We do not, however, know that the security can be replicated.

Pf: Last time: Self fin means $\underline{\Delta_n} \cdot \underline{S_{n+1}} = \Delta_{n+1} \cdot S_{n+1}$

Under $\tilde{P}$, $X \to$ self fin $\Rightarrow D_n X_n$ is a $\tilde{P}$ mg.

NTS: V_n = AFP at time n of the sec.

$\iff$ Extended market $(\underbrace{S^0 \,\&\, S^1, \dots S^d} \,\&\, \underbrace{V})$ is arb free.

new sec

By FTAP (part 1): Existence of a RNM $\Rightarrow$ No arb.

Will find a RNM for the extended market.

<u>Claim</u> $\tilde{P}$ is a RNM on the extended market!

Pf: ① Already know $D_n S_n^i$ is a $\tilde{P}$ mg $\forall i \in \{0, \dots d\}$.

② NTS $D_n V_n$ is a $\tilde{P}$ - mg

Note $V_n = \frac{1}{D_n} \tilde{E}_n (D_N V_N)$

$$\Rightarrow \underbrace{D_n V_n = \tilde{E}_n (D_N V_N)}$$

$$\Rightarrow \tilde{E}_n (D_{n+1} V_{n+1}) = \tilde{E}_n \left(\tilde{E}_{n+1} (D_N V_N) \right)$$

$$\overset{\text{tower}}{=} \tilde{E}_n (D_N V_N) = D_n V_n \quad \text{QED.}$$

Goal: Pf of converse $\rightarrow$ No arb $\Rightarrow \exists$ a RNM.

Lemma 7.8. *Suppose the market has no arbitrage, and X is the wealth process of a self-financing portfolio. If for any n, $X_n = 0$ and $X_{n+1} \geqslant 0$, then we must have $X_{n+1} = 0$ almost surely.*

(Def of No arb: $X_0 = 0$, $X_N \geq 0 \Rightarrow X_N = 0$

(self fin)

Lemma: $X_n = 0$, $X_{n+1} \geq 0 \Rightarrow X_{n+1} = 0$

Pf: If $\exists$ an arb between time n & $n+1$, then put \$ in bank at time $n+1$ & wait until N.

Lemma 7.9. *Suppose we find an equivalent measure* $\tilde{\boldsymbol{P}}$ *such that whenever* $\boxed{\Delta_n \cdot S_n = 0,\ \text{we have } \tilde{\boldsymbol{E}}_n(\Delta_n \cdot S_{n+1}) = 0}$*, then* $\tilde{\boldsymbol{P}}$ *is a risk neutral measure.*

(Random $\Delta_n = (\Delta_n^0, \Delta_n^1, \dots \Delta_n^d)$

$$\Delta_n \cdot S_n = \sum_{i=0}^{d} \Delta_n^i S_n^i$$

① Lets check $D_n S_n^1$ is a $\tilde{P}$ mg.

NTS $\tilde{E}_n(D_{n+1} S_{n+1}^1) = D_n S_n$.

- At time n, buy 1 share of S^1 & borrow from bank i.e. $\Delta_n^1 = 1$ (1 share of S^1)

$$\Delta_n^0 = -S_n^1 \cdot \left(\frac{1}{S_n^0}\right)$$

$\Delta_n^i = 0 \quad \forall i \geq 1.$

$$\Delta_n \cdot S_n = \Delta_n^0 S_n^0 + \Delta_n^1 S_n^1 + 0$$

$$= -\frac{S_n^1}{S_n^0} S_n^0 + 1 \cdot S_n^1 + 0 = 0$$

By assumption: $\tilde{E}_n\left(\Delta_n \cdot S_{n+1}\right) = 0$

Compute $\Delta_n \cdot S_{n+1} = -\frac{S_n^1}{S_n^0} \cdot S_{n+1}^0 + 1 \cdot S_{n+1}^1 + 0$

$\Rightarrow \tilde{E}_n\left(\Delta_n \cdot S_{n+1}\right) = -\frac{S_n^1}{S_n^0} S_{n+1}^0 + \tilde{E}_n S_{n+1}^1 = 0$

$\Rightarrow \tilde{E}_n S_{n+1}^1 = S_n^1 \cdot \frac{S_{n+1}^0}{S_n^0} \Rightarrow \tilde{E}_n\left(D_{n+1} S_{n+1}^1\right) = D_n S_n^1$

$\left(\because D_n = \frac{1}{S_n^0} \;\&\; D_{n+1} = \frac{1}{S_{n+1}^0}\right)$

QED.

Lemma 7.10. *Suppose $\tilde{p}$ is a probability mass function such that $\tilde{p}(\omega) = \tilde{p}_1(\omega_1)\tilde{p}_2(\omega_1, \omega_2) \cdots \tilde{p}_N(\omega_1, \dots, \omega_N)$. If X_{n+1} is $\mathcal{F}_{n+1}$-measurable, then*

$$\tilde{\boldsymbol{E}}_n X_{n+1}(\omega) = \sum_{i=1}^{M} \tilde{p}_{n+1}(\omega', j) X_{n+1}(\omega', j) \,, \qquad \textit{where} \qquad \omega' = (\omega_1, \dots, \omega_n) \,, \omega = (\omega', \omega_{n+1}, \omega_{n+1}, \dots, \omega_N)$$

(Will remind you of this next time)

Lemma 7.11. *Define* $\bar{Q} \overset{\text{def}}{=} \{v \in \mathbb{R}^M \mid v_i \geqslant 0 \ \forall i \in \{1, \ldots, M\}\}$*, and* $\mathring{Q} \overset{\text{def}}{=} \{v \in \mathbb{R}^M \mid v_i > 0 \ \forall i \in \{1, \ldots, M\}\}$*. Let* $V \subseteq R^M$ *be a subspace.*

(1) $V \cap \bar{Q} = \{0\}$ *if and only if there exists* $\hat{n} \in \mathring{Q}$ *such that* $|\hat{n}| = 1$ *and* $\hat{n} \perp V$*.*

(2) *The unit normal vector* $\hat{n} \in \mathring{Q}$ *is unique if and only if* $V \cap \bar{Q} = \{0\}$ *and* $\dim(V) = M - 1$*.*

Remark 7.12. This can be proved using the *Hyperplane separation theorem* used in convex analysis.