Lecture 30 (11/15) Please enable video if you can

7. Fundamental theorems of Asset Pricing

7.1. Markets with multiple risky assets.

(1) $\Omega = \{1, \dots, M\}^N$ is a probability space representing N rolls of M-sided dies, and p is a probability mass function on Ω.
(2) The die rolls need not be i.i.d.
(3) Consider a financial market with $d+1$ assets $S^0, S^1, \dots, S^d$. (S_n^k denotes the price of the k-th asset at time n.)
(4) For $i \in \{1, \dots, d\}$, S^i is an adapted process (i.e. S_n^i is $\mathcal{F}_n$-measurable).
(5) The 0-th asset S^0 is assumed to be a *risk free* bank/money market:
   (a) Let r_n be an adapted process specifying the interest rate at time n.
   (b) Let $S_0^0 = 1$, and $S_{n+1}^0 = (1 + r_n) S_n^0$. (Note S^0 is predictable.) in bank.
   (c) Let $D_n = (S_n^0)^{-1}$ be the discount factor (D_n dollars at time 0 becomes 1 dollar at time n).
(6) Let $\Delta_n = (\Delta_n^0, \dots, \Delta_n^d)$ be the position at time n of an investor in each of the assets $(S_n^0, \dots, S_n^d)$.
(7) The wealth of an investor holding these assets is given by $X_n = \Delta_n \cdot S_n \overset{\text{def}}{=} \sum_{i=0}^{d} \Delta_n^i S_n^i$.

$i \longrightarrow$ price of ith asset.

$n \longrightarrow$ time n.

Dot product.

$\Delta_n^0 \longrightarrow$ Cash in bank at time n

$\Delta_n^i \longrightarrow$ # shares of ith stock at time n.

Definition 7.1. Consider a portfolio whose positions in the assets at time n is Δ_n. We say this portfolio is *self-financing* if Δ_n is adapted, and $\boxed{\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}.}$

At time $n \longrightarrow$ position Δ_n

Wealth $\Delta_n \cdot S_n = \sum_{i=0}^{d} \Delta_n^i S_n^i$

Time $n+1 \longrightarrow$ Stock prices change from $S_n \longrightarrow S_{n+1}$

New wealth: $\Delta_n \cdot S_{n+1}$

↳ Change positions on the assets

$\boxed{\text{Rule}}$ → No external cash flows ($ in the market stays in the market).

New positions at time $n+1$ are $\Delta^0_{n+1}, \Delta^1_{n+1}, \ldots \Delta^d_{n+1}$

No external cash flow ⟹ Wealth should be the same

$$\Rightarrow \boxed{\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}}$$

7.2. **First fundamental theorem of asset pricing.**

Definition 7.2. We say the market is arbitrage free if for any self financing portfolio with wealth process X, we have: $X_0 = 0$ and $X_N \geqslant 0$ implies $X_N = 0$ almost surely.

Definition 7.3. We say $\tilde{\boldsymbol{P}}$ is a *risk neutral measure* if $\tilde{\boldsymbol{P}}$ is equivalent to $\boldsymbol{P}$ and $\tilde{\boldsymbol{E}}_n(D_{n+1}S^i_{n+1}) = D_n S^i_n$ for every $i \in \{0, \dots, d\}$.

Theorem 7.4. *The market defined in Section 7.1 is arbitrage free if and only if there exists a risk neutral measure.*

$\tilde{P}$ is equiv to P if when $P(A) = 0 \iff \tilde{P}(A) = 0$

Note $\tilde{P}$ does not depend on i.

i.e. $\forall i \in \{1, \dots d\}$, $\tilde{E}_n(D_{n+1} S^i_{n+1}) = D_n S^i_n$

(for $i = 0$: By def $D_n = \frac{1}{S^0_n} \iff D_n S_n = 1$)

Diff from Binom: D_n is random

(D_n is a predictable process).

Lemma 7.5. *If $\tilde{\boldsymbol{P}}$ is a risk neutral measure, then the discounted wealth of any self financing portfolio is a $\tilde{\boldsymbol{P}}$-martingale.*

Proof that existence of a risk neutral measure implies no-arbitrage.

Pf of 7.5: Say $\tilde{P}$ is a RNM

$$\Rightarrow \forall i \in \{0, \dots d\}, \quad \tilde{E}_n (D_{n+1} S^i_{n+1}) = D_n S^i_n$$

Let X_n = wealth of any self fin port

$$\Rightarrow X_n = \Delta_n \cdot S_n \quad \& \quad \Delta_n \text{ adapted}$$

$$\boxed{\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}}$$

NTS $\tilde{E}_n(D_{n+1} X_{n+1}) = D_n X_n$

Note: $\tilde{E}_n\left(D_{n+1} X_{n+1}\right) = \tilde{E}_n\left(D_{n+1} \Delta_{n+1} \cdot S_{n+1}\right)$

$$= \tilde{E}_n\left(D_{n+1} \Delta_n \cdot S_{n+1}\right) \qquad (\because X_n \text{ is self fin})$$

$$= \sum_{i=0}^{d} \Delta_n^i \, \tilde{E}_n\left(D_{n+1} S_{n+1}^i\right) \qquad (\because \Delta_n \text{ is } \mathcal{F}_n\text{-meas}).$$

$$= \sum_{i=0}^{d} \Delta_n^i \, D_n S_n^i \qquad (\because \text{Dfn of RNM})$$

$$= D_n \Delta_n \cdot S_n = D_n X_n$$

QED.

Pf that $\exists$ a RNM $\Rightarrow$ No arb!

Pf: $\tilde{P} \longrightarrow$ RNM.

Start with $X_0 = 0$ & X = wealth of a self fin port.

Suppose $X_N > 0$ NTS $X_N = 0$

Pf: Note $\tilde{E}(\underbrace{D_N X_N}) = D_0 X_0 = 0$

($\because$ $D_n X_n$ is a $\tilde{P}$ mg)

Know $D_N X_N > 0$ $\underbrace{\tilde{P} \text{ a.s.}}$

$\Rightarrow D_N X_N = 0$ ($\tilde{P}$ a.s.) $\Rightarrow D_N X_N = 0$ P (a.s.)