Lecture 29 (11/12). Please enabel video if you can

Last time: (1) American opt intrinsic Value G

can be replicated! (i.e. $\exists$ a self fin port wealth X_n such that

(1) $X_n \geq G_n$

& (2) $X_{\tau^*} = G_{\tau^*}$ for some stopping time τ^*

Proposition 6.83. *If X is the wealth of a replicating portfolio with $X_{\sigma^*} = G_{\sigma^*}$. Then σ^* is an optimal exercise policy. Moreover, if τ^* is any optimal exercise policy, then $X_{\tau^*} = G_{\tau^*}$*

Corollary 6.84 (Uniqueness). *If X, and Y are wealth of two replicating portfolios for an American option with intrinsic value G, then for any optimal exercise time σ^* we must have $\mathbf{1}_{n \leqslant \sigma^*} X_n = \mathbf{1}_{n \leqslant \sigma^*} Y_n$.*

Recall: ① An option with payoff G_σ at time σ has AFP $\tilde{E}(D_\sigma G_\sigma)$ at time 0

② Let $V_0^\sigma = \tilde{E}(D_\sigma G_\sigma)$ = AFP of the fixed cont option that pays G_σ at time σ.

③ Optimal exercise policy: $\sigma^* \ni V_0^{\sigma^*} = \max_\sigma V_0^\sigma$

<u>Claim:</u> $X_{\tau^*} = G_{\tau^*} \Rightarrow \tau^*$ is an optimal exercise policy

<u>Pf:</u> NTS $V_0^{\tau^*} \geq V_0^{\tau} \quad \forall \tau$ (finite stopping times)

i.e. NTS $\tilde{E} D_{\tau^*} G_{\tau^*} \geq \tilde{E}(D_\tau G_\tau)$

Pf: $\tilde{E}(D_{\tau^*} G_{\tau^*}) = \tilde{E}(D_{\tau^*} X_{\tau^*}) \overset{OST}{=\!=} \tilde{E}(D_0 X_0) \overset{OST}{=} \tilde{E}(D_\tau X_\tau)$

$\geq \tilde{E}(D_\tau G_\tau) = V_0^{\tau}$ QED

Conversely: Say τ^* is an optimal exercise policy

Then NTS $X_{\tau^*} = G_{\tau^*}$.

Pf: $$\tilde{E}\left(D_{\tau^*} G_{\tau^*}\right) = V_0^{\tau^*} = \max_{\sigma} V_0^{\sigma} = V_0^{\sigma^*} = \tilde{E}\left(D_{\sigma^*} G_{\sigma^*}\right)$$
$$= \tilde{E}\left(D_{\sigma^*} X_{\sigma^*}\right)$$

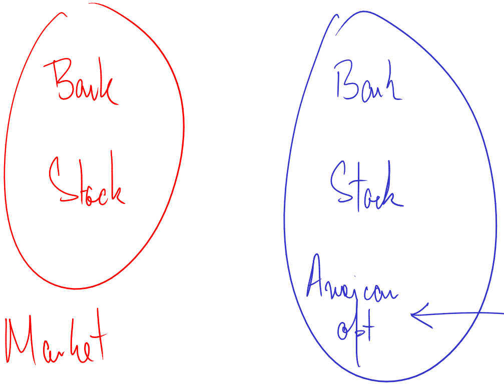
$$\Rightarrow \tilde{E}\left(D_{\tau^*} G_{\tau^*}\right) = \tilde{E}\left(D_{\sigma^*} X_{\sigma^*}\right) \overset{OST}{=} \tilde{E}\left(D_0 X_0\right) \overset{OST}{=} \tilde{E}\left(D_{\tau^*} X_{\tau^*}\right)$$

$$\Rightarrow D_{\tau^*} G_{\tau} = D_{\tau^*} X_{\tau^*}$$ ($\because$ $D_{\tau^*} G_{\tau^*} \leq D_{\tau^*} X_{\tau^*}$ & $\tilde{E}$ are equal).

QED!

Proposition 6.85. *Let $V_N = G_N$, and $V_n = \max\{G_n, D_n^{-1}\tilde{E}_n V_{n+1}\}$. Then V_n is the arbitrage free price of the American option. That is, the market remains arbitrage free if we are allowed to trade an American option at price V_n.*

$$\Rightarrow V_n = \max\left\{G_n, \frac{1}{D_n}\tilde{E}_n\left(D_{n+1}V_{n+1}\right)\right\}.$$



Market

Ext market

Price V_n at time n.

NTS: Ext is arb free.

(1) Say we buy one Am opt at time n (Price V_n)

Let $0 = \underbrace{V_n}_{\substack{\text{price of} \\ \text{american opt}}} - \underbrace{Y_n}_{\text{Borrowed \$}}$ (invest in cash/stock). $(V_n = Y_n)$

Sell option at time $\tau \geq n$ ($\tau \to$ stopping time).

Wealth at time $\tau = V_\tau - Y_\tau$

$$\Rightarrow \tilde{E}_n\left(D_\tau V_\tau - D_\tau Y_\tau\right) = \tilde{E}_n(D_\tau V_\tau) - D_n Y_n$$

($\because$ Y - self fin $\Rightarrow$ $D_n Y_n$ is a $\tilde{P}$ mg).

(Note $V_n = \max\left\{G_n, \frac{1}{D_n} E_n\left(D_{n+1} V_{n+1}\right)\right\}$

$\Rightarrow D_n V_n \geq \tilde{E}_n\left(D_{n+1} V_{n+1}\right)$

(i.e. $D_n V_n$ is a super mg under $\tilde{P}$))

$$\hookrightarrow \;\Rightarrow\; \tilde{E}_n\left(D_\tau V_\tau - D_\tau Y_\tau\right) = \tilde{E}_n\left(D_\tau V_\tau\right) - D_n Y_n$$

$$\leq D_n V_n - D_n Y_n \qquad \left(\because D_n V_n \text{ is a } \tilde{P} \text{ supermg}\right)$$

$$\leq 0$$

$\Rightarrow$ no arb is possible!

② Say we sell an Am opt at time n.

$$\boxed{0 = \underbrace{Y_n}_{\text{invest in cash / stock.}} - \underbrace{V_n}_{\text{price of an opt}}}$$

(Client exercises American option optimaly (at time $\overbrace{\tau^* \vee n}^{\max\{\tau^*, n\}}$).

$$D_n V_n = D_n X_n - A_n \qquad (\text{Doob decomposition})$$

Know $A_{\tau^*} = 0$

Let $\tau = \tau^* \vee n$ & note

$$D_\tau V_\tau - D_n V_n = D_\tau X_\tau - D_n X_n - (\underbrace{A_\tau - A_n}_{0})$$

$$= D_\tau X_\tau - D_n X_n$$

$$\Rightarrow \tilde{E}_u\left(\underline{D_\tau} V_\tau\right) = \tilde{E}_u\left(D_u V_u\right) = \underline{D_u V_u}$$

$$\Rightarrow \tilde{E}_u\left(D_\tau Y_\tau - D_\tau V_\tau\right) = D_u Y_u - D_u V_u = 0$$

No arb is possible!

Q.E.D.