Lecture 28 (11/10) Please enable video if you can

Last time: Reward Process $G \leftarrow$

$$V_N = G_N \quad \& \quad V_n = \max\{G_n, E_n V_{n+1}\}$$

$$\tau^* = \min\{\forall n \mid V_n = G_n\}.$$

Showed V solves the optimal stopping problem

& τ^* = smallest optimal stopping time

Theorem 6.81. *Let $V = \overset{X}{\cancel{M}} - A$ be the Doob decomposition for V, and define $\tau^* = \max\{n \mid A_n = 0\}$. Then τ^* is a stopping time and is the* largest *solution to the optimal stopping problem for G.*

i.e. $E G_{\tau^*} \geq E G_\sigma$ for any finite stopping time σ.

$X \to M_g$

$A \to$ 'Pred inc, $A_0 = 0$

Pf: (1) Check τ^* is a stopping time.

NOTE: In general $\max\{n \mid Y_n = b\}$ is NOT a stopping time if Y is adapted.

But for us: A is pred & inc & this makes τ^* a stopping time.

Note : $\{\tau^* = n\} = \underbrace{\underbrace{\{A_n = 0\}}_{\mathcal{F}_{n-1}\text{-meas}} \cap \underbrace{\{A_{n+1} > 0\}}_{\mathcal{F}_n\text{-meas}}}_{\mathcal{F}_n\text{-meas.}}$ ($\because$ A is inc)

(Recall: Predictable means

A_{n+1} is $\mathcal{F}_n$-meas)

2 Claim: $X_{\tau^*} = G_{\tau^*} = V_{\tau^*}$

Pf: ① $X_{\tau^*} = V_{\tau^*} - A_{\tau^*}$ & $A_{\tau^*} = 0 \Rightarrow X_{\tau^*} = V_{\tau^*}$.

② NTS $V_{\tau^*} = G_{\tau^*}$.

Say $\tau^* = n$. (Consider the event $\{\tau^* = n\}$)

$$V_{n+1} = X_{n+1} - A_{n+1}$$

$$\mathbb{1}_{\{\tau^*=n\}} E_n V_{n+1} = \mathbb{1}_{\{\tau^*=n\}} \Big(X_n - \underbrace{A_{n+1}}_{>0} \Big)$$

$\Rightarrow$ On $\{\underline{\tau}^* = n\}$, $E_n V_{n+1} < X_n = V_n + A_n = V_n + 0.$

$\Rightarrow$ On $\{\tau^* = n\}$, $E_n V_{n+1} < V_n$

Know $\underline{V_n} = \max\{\underline{G_n}, \underbrace{E_n V_{n+1}}\} \Rightarrow$ On $\underline{\{\tau^* = n\}}$, $\underline{V_n} = G_n$.

Since this holds $\forall n$ $\Rightarrow$ $V_{\tau^*} = G_{\tau^*}$.

QED (Claim 2).

Claim 3 τ^* is a soln to the optimal stopping problem.

Pf: Note for any stopping time τ,

$$E G_{\tau^*} = E X_{\tau^*} \overset{OST}{=} X_0 \overset{OST}{=} E X_\tau \geq E G_\tau$$

QED.

$$\boxed{\begin{array}{l} V = X - A,\ V \geq G \\ \Rightarrow X \geq V \geq G \end{array}}$$

Claim 4: τ^* is the largest solution to the optimal stopping problem.

Say τ^* is any solution to the optimal stopping problem

$$\Rightarrow \mathbb{E}\, G_{\tau^*} = \max_{\tau} \mathbb{E}\, G_{\tau} = \mathbb{E}\, G_{\tau^*} = \mathbb{E}\, X_{\tau^*}.$$

$$\Rightarrow \mathbb{E} G_{\tau^*} = \mathbb{E} X_{\tau^*} \overset{OST}{=} X_0 \overset{OST}{=} \mathbb{E} X_{\tau^*}.$$

$$\Rightarrow G_{\tau^*} = X_{\tau^*} \quad (\because X_{\tau^*} \geq G_{\tau^*} \;\&\; \mathbb{E} X_{\tau^*} = \mathbb{E} G_{\tau^*}).$$

Know $\quad X_\cdot \geq V \geq G \quad \Rightarrow \quad X_{\sigma^*} = V_{\sigma^*} = G_{\sigma^*}$

$$\Rightarrow \quad A_{\sigma^*} = 0 \quad \Rightarrow \quad \underline{\sigma^* \leq \tau^*}$$

(def of τ^*)

QED.

6.7. **American options (with proofs).** Consider the N period binomial model with $0 < d < 1 + r < u$.

Proposition 6.82. *Any American option can be replicated. That is, consider an American option with intrinsic value G. There exists a self financing portfolio X such that:*

1. *(1) $X_n \geqslant G_n$ for all n*
2. *(2) For some stopping time σ^*, we have $X_{\sigma^*} = G_{\sigma^*}$.*

Pf: Let $\tilde{P}$ be the RNM.

Recall X is self financing $\iff$ $D_n X_n$ is a $\tilde{P}$ mg

$$\left(D_n = (1+r)^{-n} \right).$$

Let $V_N = G_N$ & $V_n = \max\left\{ G_n , \frac{1}{D_n} \tilde{E}_n \left(D_{n+1} V_{n+1} \right) \right\}$

$$\tau^* = \min\{n \mid V_n = G_n\}.$$

Snell: $D_n V_n$ is the smallest super mg & $V_n \geq G_n$ $\forall n$.

(& τ^* is the smallest ~~soln to the optimal stopping problem.~~ optimal exercise policy).

Doob decompose $D_n V_n$: Write

$$D_n V_n = \underbrace{D_n X_n}_{\tilde{P}\text{-Mg}} - \underbrace{A_n}_{\text{Pred, inc } A_0 = 0}$$

X_n = wealth of a self fin Port ($\because D_n X_n$ is a $\tilde{P}$ mg).

Also, $D_n X_n = D_n V_n + A_n \geq D_n V_n \geq D_n G_n$

$\Rightarrow X_n \geq G_n \Rightarrow$ cond 1

Finally: $V_{\tau^*} = X_{\tau^*} = G_{\tau^*}$ (Snell)

$\Rightarrow$ Done

QED.

Proposition 6.83. *If X is the wealth of a replicating portfolio with* $\boxed{X_{\sigma^*} = G_{\sigma^*}}$*. Then σ^* is an optimal exercise policy. Moreover, if τ^* is any optimal exercise policy, then $X_{\tau^*} = G_{\tau^*}$*

Corollary 6.84 (Uniqueness)**.** *If X, and Y are wealth of two replicating portfolios for an American option with intrinsic value G, then for any optimal exercise time σ^* we must have $\mathbf{1}_{n \leqslant \sigma^*} X_n = \mathbf{1}_{n \leqslant \sigma^*} Y_n$.*

Pf: Know $D_n X_n$ is a $\tilde{P}$ - mg

$X_n \geq G_n$ & $X_{\tau^*} = G_{\tau^*}$.

NTS. $V^{\tau^*} = \max_{\tau^*} V_0^{\tau}$ $\left(V_0^{\tau} = \tilde{E}\left(D_\tau G_\tau\right)\right)$.