Lecture 27 (11/8). Please enable video if you can

Last time: Optimal stopping problem

Play game. Can leave at any time $\longrightarrow$ reward G_n.

Stop at τ $\longrightarrow$ Get reward G_τ

Goal: $\max_\tau E\, G_\tau$

① Let $V_N = G_N$ & $V_n = \max\{\underset{\text{stop}}{G_n},\ \overbrace{E_n V_{n+1}}\}$ $\longleftarrow$

Snell: V is the smallest super mg $\ni$ $V_n \geq G_n$ $\forall n$.

(2) Let W be a mg, $W_n \geq G_n$ $\forall n$.

and for some stopping time τ^*, $E W_{\tau^*} = E G_{\tau^*}$

Then $\underbrace{W_{\tau^* \wedge n}}_{\text{mg}} = V_{\tau^* \wedge n}$ $\left(\iff \forall n, \ \mathbb{1}_{\{n \leq \tau^*\}} W_n = \mathbb{1}_{\{n \leq \tau^*\}} V_n \right)$

(Last time).

(I.e. V is a mg before time τ^*).

Theorem 6.78. *Let $\sigma^* = \min\{n \mid V_n = G_n\}$. Then σ^* is the minimal solution to the* optimal stopping problem *for G. Namely, $\boldsymbol{E}G_{\sigma^*} = \max_\sigma \boldsymbol{E}G_\sigma$ where the maximum is taken over all finite stopping times σ. Moreover, if $\boldsymbol{E}G_{\tau^*} = \max_\sigma \boldsymbol{E}G_\sigma$ for any other finite stopping time τ^*, we must have $\tau^* \geqslant \sigma^*$.*

Remark 6.79. By construction $V_{\sigma^* \wedge n}$ is a martingale.

Pf: Last time: V is a super mg

Write $\underset{\sim}{V} = \underset{\text{mg}}{\underbrace{X}} - \underset{\text{Predictable, increasing, } A_0 = 0}{\underbrace{A}}$

Claim 1 (Last time): $A_{\tau^*} = 0$

Pf: $V_n = \max\{G_n, \underline{E_n V_{n+1}}\}$ $\boxed{\tau^* = \min\{n \mid V_n = G_n\}}$

$\Rightarrow$ If $n < \tau^*$, then $V_n \neq G_n$

i.e. If $n < \tau^*$ $V_n = E_n V_{n+1}$

Now note $E_n V_{n+1} = E_n X_{n+1} - E_n A_{n+1}$ $\quad (n \underline{\underline{<}} \tau^*)$

$\Rightarrow$ $V_n = X_n - \underset{\approx}{A_{n+1}}$ $\quad$ ($\because$ A is pred
X is a mg
& $n < \tau^* \Rightarrow E_n V_{n+1} = V_n$)

Already know $V_n = X_n - A_n$

$\Rightarrow \forall n < r^*$ must have $A_{n+1} = A_n$.

Since $A_0 = 0 \Rightarrow A_{n+1} = A_n = A_{n-1} \dots = A_0 = 0$.

$\forall n < r^*$

i.e $\forall m \leq r^*$, must have $A_m = 0 \Rightarrow$ Claim

QED

② $\Rightarrow$ $\underline{X}_{\tau^*} = \underline{V}_{\tau^*} = \underline{G}_{\tau^*}$ $\left(\because V_{\tau^*} = X_{\tau^*} - \underbrace{A_{\tau^*}}_{0} \right.$ /

& def of $\left. \tau^* \right)$.

③ Claim: τ^* is a soln to the optimal stopping prob for G.

i.e. $\forall$ fin stopping times τ, $E G_\tau \leq E G_{\tau^*}$

Pf: Note $G \leq V = X - A$ $\Rightarrow$ $X \geq V \geq G$.

$$\Rightarrow \quad E G_\tau \leq E X_\tau \overset{OST}{=} X_0 \overset{OST}{=} E X_{\sigma^*} = E G_{\sigma^*}.$$

QED.

④ <u>Claim</u>: If τ^* is any soln to the optimal stopping problem then $\tau^* \geq \sigma^*$.

Pf: Chose τ^* to be any soln to the optimal stopping problem.

i.e. $E G_{\tau^*} = \max_\tau E G_\tau$.

NTS $\boxed{\tau^* \geq \sigma^*}$.

Claim: $X_{\tau^*} = V_{\tau^*} = G_{\tau^*}$!

Pf: $\mathbb{E} G_{\tau^*} \overset{\text{know}}{=} \max_{\sigma} \mathbb{E} G_{\sigma} \overset{\text{know}}{=} \mathbb{E} G_{\sigma^*}$

Know $\mathbb{E} G_{\tau^*} \leq \mathbb{E} X_{\tau^*}$ $(\because X \geq V \geq G)$

$\overset{\text{OST}}{=} X_0 \overset{\text{OST}}{=} \mathbb{E} X_{\sigma^*} = \mathbb{E} G_{\sigma^*} = \mathbb{E} G_{\tau^*}$

$\Rightarrow \mathbb{E} G_{\tau^*} = \mathbb{E} X_{\tau^*}$

$\Rightarrow G_{\tau^*} = X_{\tau^*}$ $\quad (\because X_{\tau^*} \geq G_{\tau^*})$

Since $X \geq V \geq G$ $\Rightarrow$ $X_{\tau^*} = V_{\tau^*} = G_{\tau^*}$ $\Rightarrow$ claim.

This implies $\tau^* \geq \sigma^*$ ($\because$ $\sigma^* =$ first time $V = G$ & $V_{\tau^*} = G_{\tau^*}$)

QED

Theorem 6.80. *For any $k \in \{0, \ldots, N\}$, let $\sigma_k^* = \min\{n \geqslant k \mid V_n = G_n\}$. Then $\boldsymbol{E}_k G_{\sigma_k^*} = \max_{\sigma_k} \boldsymbol{E}_k G_{\sigma_k}$, where the maximum is taken over all finite stopping times σ_k for which $\sigma_k \geqslant k$ almost surely.*

(Pf: You check).