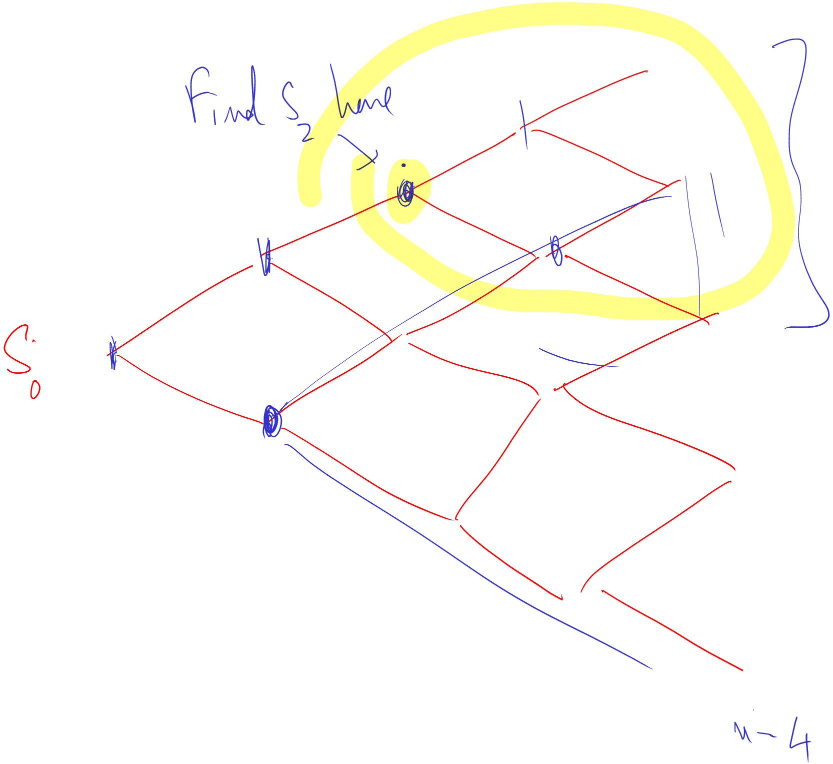
Find S_2 here
S_0
$n=4$

Q2] American option

$g_n(S_n) = G_n$ = intrinsic val at time n

Every exercise policy is optimal.

Find a nice val for g_n.

Exercise policy τ^* optimal means $V_0^{\tau^*} = \max_\tau V_0^\tau$

$\tau^* \longrightarrow$ finite stopping time

$V_0^\tau =$ AFP at time 0 of an security that matures at time $\underline{\tau}$ & pays $\underline{G_\tau}$

Guess: Every exercise policy is optimal $\Rightarrow$ $\overset{G_n}{\cancel{V_n}} = \frac{1}{D_n} \tilde{E}_n D_{n+1} V_{n+1}$

Assume guess: Know $g_n(S_n) = \max\left\{ \underline{g_n(S_n)} , \underline{\frac{1}{1+r} \tilde{E}_n g_{n+1}(S_{n+1})} \right\}$

Should give $g_n(S_n) = \frac{1}{1+r} \tilde{E}_n \, g_{n+1}(S_{n+1})$

$$g_n(s) = \frac{1}{1+r}\left[g_{n+1}(us)\,\tilde{p} + g_{n+1}(ds)\,\tilde{q} \right]$$

Pf that this should work:

V_n = AFP of American option

V_0^τ = AFP " security that pays G_τ at time τ.

$$= \tilde{E}\left(D_\tau G_\tau\right)$$

$$\left(V_n^\tau \mathbb{1}_{\{n \leq \tau\}} = \frac{1}{D_n} \tilde{E}_n\left(D_\tau G_\tau\right) \mathbb{1}_{\{n \leq \tau\}}.\right)$$

Given every exercise policy is optimal.

$$\iff V_0^{\sigma^*} = \max_\tau V_0^\tau$$

$$\iff \text{For any 2 finite stopping times } V_0^\tau = V_0^{\tau'}$$

$$\iff \quad \tilde{E}\left(D_\tau G_\tau\right) = G_0 \qquad \forall \text{ stopping times } \tau.$$

HW 7 Q4 $\Rightarrow$ $D_n G_n$ is a mg (under $\tilde{P}$)

$$\Rightarrow \underbrace{\underset{\overset{\|}{g_n(S_n)}}{G_n} = \frac{1}{D_n}\tilde{E}_n\left(D_{n+1}G_{n+1}\right) = \frac{1}{1+r}\tilde{E}_n\, g_{n+1}\left(S_{n+1}\right)}_{\text{Was what I guessed.}}$$

~~#~~ HW 9 Q 5

(a) $G \to$ intrinsic val

$$\underline{X_n \geq G_n}, \quad X_n \text{ self fin.}$$
$$X_{\tau^*} = G_{\tau^*}$$

<u>Claim</u>: $X_0 = \max_{\tau} V_0^{\tau}$

(Recall $V_0^{\tau} = \tilde{E}(D_\tau G_\tau)$)

Pf: Claim 1: $X_0 \geq V_0^\sigma \quad \forall \sigma$

Pf: Note $V_0^\sigma = \tilde{E}(D_\sigma G_\sigma) \leq \tilde{E}(D_\sigma X_\sigma) \stackrel{OST}{=} X_0$

QED.

Claim 2: $X_0 = V_0^{\sigma^*}$

(Pf: $V_0^{\sigma^*} = \tilde{E}(D_{\sigma^*} G_{\sigma^*}) = \tilde{E}(D_{\sigma^*} X_{\sigma^*}) \stackrel{OST}{=} X_0$)

5b

American option

Ref part 1

$$X_n \geq G_n$$

$$X_{\tau^*} = G_{\tau^*}$$

X self fin

Ref part 2.

$$Y_n \geq G_n$$

$$Y_{\tau^*} = G_{\tau^*}$$

Y self fin.

(τ^* ~~t~~ need not $= \tau^*$)

Claim: $\boxed{X_{\tau^*} = Y_{\tau^*}}$ (& $\boxed{X_{\sigma^*} = Y_{\sigma^*}}$)

(& $\forall n \le \tau^* \vee \sigma^*$, $X_n = Y_n$

More correctly $X_n \mathbb{1}_{\{\sigma^* \vee \tau^* \ge n\}} = Y_n \mathbb{1}_{\{\sigma^* \vee \tau^* \ge n\}}$)

Pf of Claim: Want $X_{\tau^*} = Y_{\tau^*}$

Knows $Y_n \ge G_n$, $Y_{\tau^*} = G_{\tau^*}$ & $\underbrace{\tilde{E}(D_\sigma G_\sigma) \le \tilde{E}(D_{\tau^*} Y_{\tau^*})}$

Know $X_n \geq G_n$, $X_{\sigma^*} = G_{\sigma^*}$ & $\tilde{E}(D_\sigma G_\sigma) \leq \tilde{E}(D_{\sigma^*} X_{\sigma^*})$

$$\tilde{E}(D_{\tau^*} X_{\tau^*}) \overset{OST}{=} \tilde{E}(D_0 X_0) \overset{OST}{=} \underbrace{\tilde{E}(D_{\sigma^*} X_{\sigma^*})} = \underbrace{\tilde{E}(D_{\sigma^*} G_{\sigma^*})} \leq \underbrace{\tilde{E}(D_{\tau^*} Y_{\tau^*})}$$

$$\underbrace{\tilde{E}(D_{\tau^*} Y_{\tau^*})} = \tilde{E}(D_{\tau^*} G_{\tau^*}) \leq \tilde{E}(D_{\tau^*} X_{\tau^*})$$

$$\Rightarrow \tilde{E}(D_{\tau^*} Y_{\tau^*}) = \underbrace{\tilde{E}(D_{\sigma^*} G_{\sigma^*})} = \underbrace{\tilde{E}(D_{\sigma^*} X_{\sigma^*})} \leftarrow (*)$$

<u>Claim</u>: $\Rightarrow \quad Y_{\tau^*} = G_{\tau^*} \quad (= X_{\tau^*})$

Pf: $\tilde{E}(D_{\tau^*} Y_{\tau^*}) \overset{OST}{=} \tilde{E}(D_{\tau^*} Y_{\tau^*}) \overset{*}{=} \tilde{E}(D_{\tau^*} X_{\tau^*}) = \tilde{E}(D_{\tau^*} G_{\tau^*})$

$$\Rightarrow D_{\tau^*} Y_{\tau^*} = D_{\tau^*} G_{\tau^*} \qquad (\because D_{\tau^*} Y_{\tau^*} \geq D_{\tau^*} G_{\tau^*})$$

QED

HW 4: 3a)

$f_n(S_n)$ is a mg $\Rightarrow$ $\boxed{f_n(s) = p\, f_{n+1}(us) + q\, f_{n+1}(ds)}$

3b) Given g: Want to find $E\, g(S_N)$ (through recurrence relations)

Use (a): Chose $f_N(s) = g(s)$

Chose f_n so that $f_n(s) = p\, f_{n+1}(us) + q\, f_{n+1}(ds)$

By (a) $f_n(S_n)$ is a mg

$$\boxed{\Rightarrow \underline{f_0(S_0)} = E\, f_N(S_N) = \underline{E\, g(S_N)}}$$

(3c) $\overline{g_N(s) = s^\alpha = f_N(s)}$

$$f_{N-1}(s) = p\, f_N(us) + q\, f_N(ds) = p\, u^\alpha s^\alpha + q\, d^\alpha s^\alpha.$$

$$= \underline{(p u^\alpha + q d^\alpha)}\ \underline{\underline{s^\alpha}}$$

$$\mathbb{E}\left(S_N^{\alpha}\right) = \mathbb{E}\, g(S_N) = \underset{\omega}{g_\gamma}(S_\gamma)$$