Lecture 26 (11/1). Please enable video if you can.

Last time: Super Mg: $M_n \geq E_n M_{n+1}$

Sub Mg: $M_n \leq E_n M_{n+1}$

Doob Decomp: $X = \underbrace{M}_{\text{Mg}} + \underbrace{A}_{\substack{\text{Predictable} \\ A_0 = 0}}$ (A_{n+1} is $\mathcal{F}_n$-meas)

Cor: X is a super Mg $\Rightarrow$ $X = \underbrace{M}_{\text{Mg}} - \underbrace{A}_{\substack{\text{Predictable} \\ A_0 = 0}}$ & increasing

$(\Rightarrow E_n X_\tau \leq X_{\tau \wedge n})$

Theorem 6.76 (Snell). *Let G be an adapted process, and define V by*

$$V_N = G_N \qquad V_n = \max\{\boldsymbol{E}_n V_{n+1}, G_n\}.$$

Then V is the smallest super-martingale for which $V_n \geqslant G_n$.

Game: can stop playing a game at any finite stopping time τ

collect reward G_τ

Pf: NTS ① V is a super Mg

② $V_n \geq G_n$

& ③ If W is any super Mg + $W \geq G$ then $W \geq V$.

2 : $V_n = \max\{G_n, E_n V_{n+1}\} \Rightarrow V_n \geq G_n$.

1 : $V_n = \max\{G_n, E_n V_{n+1}\} \Rightarrow V_n \geq E_n V_{n+1}$

$\Rightarrow V_n$ is a super mg.

Pf of 3 : Let W be any super Mg. & $W \geq G$.

NTS $W \geq V$

Pf: Backward induction : 1 Certainly $W_N \geq G_N = V_N$ ✓

② Assume $W_{n+1} \geq V_{n+1}$

ⓐ $W_n \geq E_n W_{n+1}$ ($\because$ W is a super mg)

$\geq E_n V_{n+1}$ (induction Hyp)

ⓑ Already know $W_n \geq G_n$.

ⓐ & ⓑ $\Rightarrow$ $W_n \geq \max\{G_n, E_n V_{n+1}\} = V_n$ QED.

Proposition 6.77. *If W is any martingale for which $W_n \geqslant G_n$, and for one stopping time τ^* we have $\boldsymbol{E}W_{\tau^*} = \boldsymbol{E}G_{\tau^*}$, then we must have $W_{\tau^* \wedge n} = V_{\tau^* \wedge n}$, and $V_{\tau^* \wedge n}$ is a martingale.*

Pf: Note ① : $W_{\tau^*} = G_{\tau^*}$ $\quad$ ($\because$ $E W_{\tau^*} = E G_{\tau^*}$ & $W_{\tau^*} \geqslant G_{\tau^*}$)

Note ② : $W \geqslant \underline{V} \geqslant \underline{G}$ $\quad$ ($\because$ W a mg $\Rightarrow$ W is a super mg. & $W \geqslant G$ $\Rightarrow$ $W \geqslant V$)

smallest super mg $\geqslant G$.

Know: $W_{\tau^*} = G_{\tau^*} \Rightarrow W_{\tau^*} = V_{\tau^*} = G_{\tau^*}$.

$$\Rightarrow \quad W_{\tau^*} = V_{\tau^*}$$

(D.D. + OST $\to$ last time).

$$\Rightarrow \quad W_{\tau^* \wedge n} \overset{OST}{=} E_n W_{\tau^*} = E_n V_{\tau^*} \le V_{\tau^* \wedge n}$$

Since we already know $W \geq V \Rightarrow W_{\tau^* \wedge n} = V_{\tau^* \wedge n}$.

Also $V_{\tau^* \wedge n}$ is a mg because $W_{\tau^* \wedge n}$ is a mg

$$\left(OST \Rightarrow E_n\left(W_{\tau^* \wedge (n+1)}\right) = W_{\tau^* \wedge (n+1) \wedge n} = W_{\tau^* \wedge n} \right)$$

QED.

Theorem 6.78. *Let $\sigma^* = \min\{n \mid V_n = G_n\}$. Then σ^* is the minimal solution to the optimal stopping problem for G. Namely, $\boldsymbol{E}G_{\sigma^*} = \max_\sigma \boldsymbol{E}G_\sigma$ where the maximum is taken over all finite stopping times σ. Moreover, if $\boldsymbol{E}G_{\tau^*} = \max_\sigma \boldsymbol{E}G_\sigma$ for any other finite stopping time τ^*, we must have $\tau^* \geqslant \sigma^*$.*

Remark 6.79. By construction $V_{\sigma^* \wedge n}$ is a martingale.

Pf of Thm: Know V is a super Mg.

Doob decomposition: $V = \underbrace{X}_{\text{Mg}} - \underbrace{A}_{\text{Pred, inc}}$, $A_0 = 0$.

<u>Claim:</u> $A_{\sigma^*} = 0$

(Note $\Rightarrow A_{\sigma^* \wedge n} = 0 \Rightarrow V_{\sigma^* \wedge n} = \underbrace{X_{\sigma^* \wedge n}}_{\text{Mg}}$)

Pf of claim: $V_n = \max\{G_n, \underbrace{E_n V_{n+1}}\}$

$$\tau^* = \min\{n \mid V_n = G_n\}$$

$\Rightarrow$ for $n < \tau^*$, $V_n \neq G_n$ i.e. $V_n = E_n V_{n+1}$

More precisely $\mathbb{1}_{\{n<\tau^*\}} V_n = \mathbb{1}_{\{n<\tau^*\}} E_n V_{n+1}$

Know $E_n\left(\mathbb{1}_{\{n<\tau^*\}} V_{n+1}\right) = E_n\left(\mathbb{1}_{\{n<\tau^*\}} X_{n+1} - \mathbb{1}_{\{n<\tau^*\}} A_{n+1}\right)$

$$\Rightarrow \mathbb{1}_{\{n<\tau^*\}} E_n V_{n+1} = \mathbb{1}_{\{n<\tau^*\}} E_n X_{n+1} - \mathbb{1}_{\{n<\tau^*\}} E_n A_{n+1}$$

$$\Rightarrow \mathbb{1}_{\{n<\tau^*\}} V_n = \mathbb{1}_{\{n<\tau^*\}} X_n - \mathbb{1}_{\{n<\tau^*\}} A_{n+1}$$

$$(V = X - A) \Rightarrow \mathbb{1}_{\{n<\tau^*\}} A_{n+1} = \mathbb{1}_{\{n<\tau^*\}} A_n$$

$A_0 = 0 \quad \Rightarrow \quad A_{\sigma^*} = 0$

$\left(\& \ \mathbb{1}_{\{n \le \sigma^*\}} A_n = 0 \right).$

QED (Claim).

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$f(0) = 0$

$f(M) = 1$

$S_\tau \in \{0, M\}$

$E f(S_\tau) = P(S_\tau = 0) \cdot \underbrace{f(0)}_{0} + P(S_\tau = M) \underbrace{f(M)}_{1}$

$= P(S_\tau = M)$