Lecture 25 (10/29). Please enable video if you can

Rem Have not yet proved

$$\underline{\underline{V_n}} = \max \left\{ G_n , \overbrace{\frac{1}{D_n} \tilde{E}_n \left(D_{n+1} V_{n+1} \right)} \right\}$$

Gives the AFP of an american option (IOU)

6.6. **Optimal Stopping.**

Definition 6.68. We say an adapted process M is a *super-martingale* if $\boldsymbol{E}_n M_{n+1} \leqslant M_n$. (almost surely $\forall n$)

Definition 6.69. We say an adapted process M is a *sub-martingale* if $\boldsymbol{E}_n M_{n+1} \geqslant M_n$. (" " $\forall n$)

Example 6.70. The discounted arbitrage free price of an American option is a *super-martingale* under the risk neutral measure.

Mg : the fn $n \longmapsto EM_n$ is constant

Super mg : The fn $n \longmapsto EM_n$ is a decreasing fn of n

Sub mg : " " $n \longmapsto EM_n$ is an inc fn of n.

(Pf: If M is a super mg : $E_n M_{n+1} \leq M_n \Rightarrow E\underbrace{(E_n M_{n+1})}_{EM_{n+1}} \leq EM_n$

QED.

Theorem 6.71 (Doob decomposition). *Any adapted process can be uniquely expressed as the sum of a martingale and a predictable process that starts at 0. That is, if X is an adapted process there exists a unique pair of process M, A such that M is a martingale, A is predictable, $A_0 = 0$ and $X = M + A$.*

Mg

Predictable, & $A_0 = 0$

Recall: A is a predictable process if A_n is $\mathcal{F}_{n-1}$-measurable.

(In finance: Cash in bank $\rightarrow$ predictable process.)

Scratch work: Say $X_n = \underbrace{M_n}_{Mg} + \underbrace{A_n}_{\text{Pred},\ A_0=0}$

$$\Rightarrow \quad X_{n+1} = M_{n+1} + A_{n+1}$$

$$\Rightarrow \quad E_n X_{n+1} = E_n M_{n+1} + E_n A_{n+1}$$

$$E_n X_{n+1} = M_n + A_{n+1}$$

Want $A_0 = 0$. $\quad X_0 = M_0 + \underbrace{A_0}_{=0} \; \Big\} \Rightarrow X_0 = \underline{M_0}$

$$\boxed{X_1 = M_1 + A_1}$$

$$E X_1 = M_0 + \underline{A_1} \Rightarrow A_1 = E X_1 - M_0$$

(Know M_0, $A_0 = 0$ & A_1)

Since $X_1 = M_1 + A_1 \Rightarrow \boxed{M_1 = A_1 - X_1}$

Know $X_2 = M_2 + A_2 \Rightarrow \underline{E_1 X_2} = \underline{M_1} + \underline{A_2}$

$$\Rightarrow \boxed{A_2 = E_1 X_2 - M_1}$$

$$\Rightarrow M_2 = X_2 - (E_1 X_2) + M_1$$

Induction: ① Define $A_0 = 0$ & $M_0 = X_0$

② Given M_n & A_n, Define M_{n+1} & A_{n+1} by

ⓐ $A_{n+1} = E_n X_{n+1} - M_n$

ⓑ $M_{n+1} = M_n + X_{n+1} - E_n X_{n+1}$

③ Check: ⓐ Clearly A is predictable ($\because E_n X_{n+1} - M_n$ is $\mathcal{F}_n$ meas)

ⓑ Clearly M is a mg ($\because E_n M_{n+1} = M_n + E_n X_{n+1} - E_n X_{n+1} = M_n$)

& (c) $M_{n+1} + A_{n+1} = M_n + X_{n+1} - E_n X_{n+1} + E_n X_{n+1} - M_n$

$$= X_{n+1}$$ QED.

Uniqueness: (Proof above also shows uniqueness since A_0 & M_0 were unique & the choice of M_{n+1} & A_{n+1} that satisfies $X_{n+1} = M_{n+1} + A_{n+1}$ & $M \to$ mg, $A \to$ pred is also unique

QED

Proposition 6.72. *If X is a super-martingale, then there exists a unique martingale M and increasing predictable process A such that $X = M - A$.*

Proposition 6.73. *If X is a sub-martingale, then there exists a unique martingale M and increasing predictable process A such that $X = M + A$.*

Pf: Say X is a super Mg (i.e. $E_n X_{n+1} \leq X_n$)

Doob decomposition: Write $X = M + \tilde{A}$ (M is a mg, $\tilde{A}$ is predictable)

Set $A = -\tilde{A}$. $\Rightarrow X = \underbrace{M}_{\text{Mg}} - \underbrace{A}_{\text{Pred}}$

NTS: A is inc.: $X_{n+1} = M_{n+1} - A_{n+1}$

m

Condition on $\mathcal{F}_n$: $E_n X_{n+1} = E_n M_{n+1} - E_n A_{n+1}$

$(E_n X_{n+1} \leq X_n)$ $X_n \geq E_n X_{n+1} = M_n - A_{n+1}$

$\Downarrow$

$X_n = M_n - A_n$

$\Rightarrow M_n - A_n \geq M_n - A_{n+1} \Rightarrow A_{n+1} \geq A_n$

$\Rightarrow A$ is increasing QED.

Corollary 6.74. *If X is a super-martingale and τ is a bounded stopping time, then* $\boxed{\boldsymbol{E}_n X_\tau \leqslant X_{\tau\wedge n}.}$

Corollary 6.75. *If X is a sub-martingale and τ is a bounded stopping time, then* $\boldsymbol{E}_n X_\tau \geqslant X_{\tau\wedge n}$.

Recall: OST: If X is a mg & τ is a bdd stopping time

then $E_n X_\tau = X_{\tau\wedge n}$

$\Rightarrow$ Pf of 6.74: Say X is a super Mg

Doob decomp: Write $X = \underset{\text{Mg}}{\underbrace{M}} - \underset{\text{Pred incr.}}{\underbrace{A}}$

$$\Rightarrow E_n X_\tau = E_n (M_\tau - A_\tau)$$

$(a \wedge b = \min\{a, b\})$

$$= M_{\tau\wedge n} - E_n A_{\tau}$$

$$\leq M_{\tau\wedge n} - E_n \left(A_{\tau\wedge n}\right)$$

$\left(\because \tau\wedge n \leq \tau \Rightarrow A_{\tau\wedge n} \leq A_{\tau} \right)$

$$= M_{\tau\wedge n} - A_{\tau\wedge n}$$

$\left(\because A_{\tau\wedge n} \text{ is } \mathcal{F}_n\text{-meas} \right)$

$$= X_{\tau\wedge n}.$$

QED

(Note $$A_{\tau\wedge n} = \sum_{k=0}^{n} \underbrace{\underbrace{\mathbb{1}_{\{\tau=k\}}}_{\mathcal{F}_k\text{ meas}}\;\underbrace{A_k}_{\mathcal{F}_k\text{-meas}}}_{\mathcal{F}_n\text{-meas}} + \underbrace{\underbrace{\mathbb{1}_{\{\tau>n\}}}_{\mathcal{F}_n\text{-meas.}}\;\underbrace{A_n}_{\mathcal{F}_n\text{-meas}}}_{\mathcal{F}_n\text{-meas.}}$$

Theorem 6.76 (Snell). *Let G be an adapted process, and define V by*

$$V_N = G_N \qquad V_n = \max\{\boldsymbol{E}_n V_{n+1}, G_n\}\,.$$

Then V is the smallest super-martingale for which $V_n \geqslant G_n$.