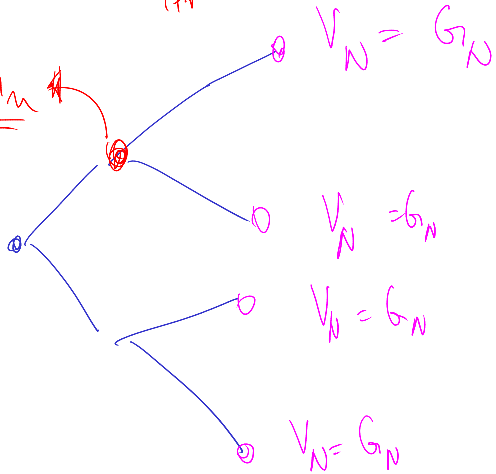
Lecture 24 (10/27): Please enable video if you can

Last time $\longrightarrow$ American options (proofs later)

Hold: Expected return: $\frac{1}{1+r}\tilde{E}_n V_{n+1} = \underbrace{\frac{1}{D_n}\tilde{E}_n(D_{n+1}V_{n+1})}$.

Exercise: G_n



Theorem 6.61. *Consider the binomial model with $0 < d < 1 + r < u$, and an American option with intrinsic value G. Define*

$$V_N = G_N\,, \qquad V_n = \max\Big\{\frac{1}{D_n}\tilde{\boldsymbol{E}}_n(D_{n+1}V_{n+1}), G_n\Big\}\,, \qquad \sigma^* = \min\{n \leqslant N \mid V_n = G_n\}\,.$$

(IOV $\to$ Pf)

Then V_n is the arbitrage free price, and σ^* is the minimal optimal exercise time. Moreover, this option can be replicated.

Remark 6.62. The above is true in any complete, arbitrage free market.

Remark 6.63. In the Binomial model the above simplifies to:

$$V_n(\omega) = \max\Big\{\frac{1}{1+r}\Big(\tilde{p}V_{n+1}(\omega', 1) + \tilde{q}V_{n+1}(\omega', -1)\Big), G_n(\omega)\Big\}\,, \qquad \text{where } \omega = (\omega', \omega_{n+1}, \omega'')\,, \quad \omega' = (\omega_1, \dots, \omega_n)\,.$$

Remark 6.64. We will prove Theorem 6.61 in the next section after proving the Doob decomposition.

Theorem 6.65. *Consider the Binomial model with $0 < d < 1 + r < u$, and a state process $Y = (Y^1, \dots, Y^d)$ such that $Y_{n+1}(\omega) = h_{n+1}(Y_n(\omega'), \omega_{n+1})$, where $\omega' = (\omega_1, \dots, \omega_n)$, $\omega = (\omega', \omega_{n+1}, \dots, \omega_N)$, and $h_0, h_1, \dots, h_N$ are N deterministic functions. Let $g_0, \dots, g_N$ be N deterministic functions, let $G_k = g_k(Y_k)$, and consider an American option with intrinsic value $G = (G_0, G_1, \dots, G_N)$. The* pre-exercise price *of the option at time n is $f_n(Y_n)$, where*

$$f_N(y) = g_N(y) \quad \text{for } y \in \operatorname{Range}(Y_N), \qquad f_n(y) = \max\Big\{g_n(y), \frac{1}{1+r}\Big(\tilde{p} f_{n+1}(h_{n+1}(y, \underset{+1}{u})) + \tilde{q} f_{n+1}(h_{n+1}(y, \underset{-1}{d}))\Big)\Big\}, \quad \text{for } y \in \operatorname{Range}(Y_n).$$

The minimal optimal exercise time is $\sigma^* = \min\{n \mid f_n(Y_n) = g_n(Y_n)\}$.

Pf: Know $V_n = \max\left\{ G_n, \frac{1}{D_n} \tilde{E}_n (D_{n+1} V_{n+1}) \right\}$

$= \max\left\{ G_n, \frac{1}{1+r} \tilde{E}_n V_{n+1} \right\}$.

(2) Backward induction: Know $V_N = G_N = g_N(Y_N)$.

Set $f_N(y) = g_N(y) \quad \forall y \in \operatorname{Range}(Y_N)$

$$(\Rightarrow V_N = G_N = f_N(Y_N)).$$

③ Say $V_{n+1} = f_{n+1}(Y_{n+1})$. NTS $V_n = f_n(Y_n)$ (find f_n).

Know $V_n = \max\left\{ G_n, \frac{1}{1+r} \tilde{E}_n V_{n+1} \right\}$.

$$= \max\left\{ g_n(Y_n), \frac{1}{1+r} \tilde{E}_n f_{n+1}(Y_{n+1}) \right\}.$$

$$= \max\left\{ g_n(Y_n), \frac{1}{1+r} \tilde{E}_n f_{n+1}(h_{n+1}(Y_n, \omega_{n+1})) \right\}$$

$$= \max\left\{ g_n(\underline{Y_n}),\ \frac{1}{1+r}\left(\tilde{p}\, f_{n+1}(h_{n+1}(\underline{Y_n}, +1)) + \tilde{q}\, f_{n+1}(h_{n+1}(\underline{Y_n}, -1))\right) \right\}.$$

Set $f_n(y) = \max\left\{ g_n(y),\ \frac{1}{1+r}\left(\tilde{p}\, f_{n+1}(h_{n+1}(y, +1)) + \tilde{q}\, f_{n+1}(h_{n+1}(y, -1))\right)\right\}$

& get $V_n = f_n(Y_n)$

Also know $\tau^* = \min\{ n \mid V_n = G_n \}$ QED.

Theorem 6.66. *Suppose the interest rate $\boxed{r \text{ is nonnegative.}}$ Let g be a convex function with $\boxed{g(0) = 0,}$ and let $G_n = g(S_n)$. Consider an American option with intrinsic value $G_n = g(S_n)$. Then $\sigma^* = N$ is an optimal exercise time. That is, it is not advantageous to exercise this option early.*

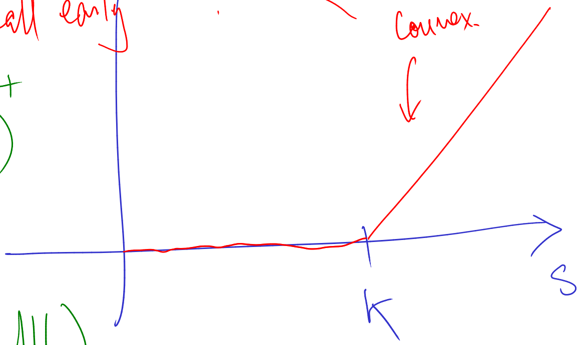
Corollary 6.67. *The arbitrage free price of an American call and European call are the same.*

Intrinsic value of American call : $g(S_n) = (S_n - K)^+$

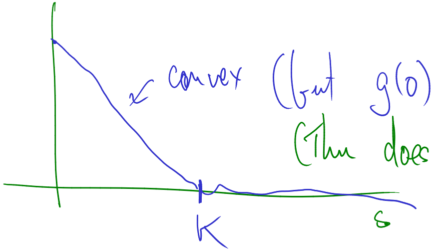
& $g(0) = 0$

Thm $\Rightarrow$ No advantage to exercising American call early

Convex



Intrinsic val of American Put : $g(S_n) = (K - S_n)^+$

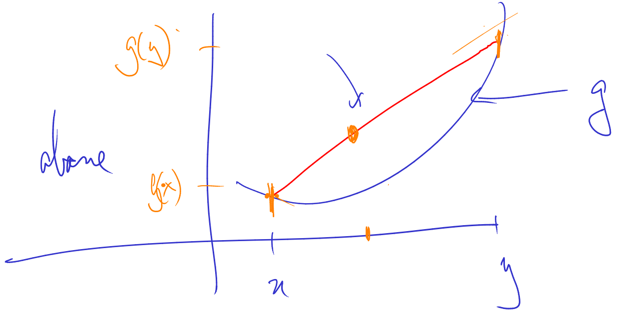


convex (but $g(0) \neq 0$)

(Thm does not apply).

Pf of 6.66 : (1) Convex :

By def → chord lies above the fn.



⟺ (2) ∀ x, y ∈ Domain(g), θ ∈ [0, 1],

From Calc:

line ⟺

$$\underbrace{\theta g(x) + (1-\theta) g(y) \geq g(\overbrace{\theta x + (1-\theta) y}^{z})}$$

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③ Know $g(0) = 0$ & g is convex.

$\Rightarrow \forall s \geq 0$, & $\theta \in [0,1]$, $g(\theta \cdot 0 + (1-\theta) s) \leq \theta g(0) + (1-\theta) g(s)$

$\Longleftrightarrow g((1-\theta) s) \leq (1-\theta) g(s)$. $\circledast$

④ Know $V_n = \max\{ g(S_n), \underbrace{\frac{1}{1+r} \tilde{E}_n V_{n+1}} \}$.

Backward induction: Suppose $V_{n+1} \geq g(S_{n+1})$ (True for $n+1 = N$)

Today : $\frac{1}{1+r}\tilde{E}_n V_{n+1} \geq \frac{1}{1+r}\tilde{E}_n g(S_{n+1}) = \tilde{E}_n\left(\frac{1}{1+r} g(S_{n+1})\right)$

$\geq \tilde{E}_n g\left(\frac{S_{n+1}}{1+r}\right)$ (from ⊛ since $r \geq 0$

$\Rightarrow \frac{1}{1+r} \in [0,1]$

& can chose $1-\theta = \frac{1}{1+r}$]

(Jensen's Inequality) $\geq g\left(\tilde{E}_n \frac{S_{n+1}}{1+r}\right)$

$= g(S_n)$ $\left[\because \tilde{E}_n S_{n+1} = (1+r) S_n\right]$

Know $V_n = \max\left\{ g(S_n), \frac{1}{1+r}\tilde{E}_n V_{n+1} \right\}$

$$\geq \max\left\{ g(S_n), \underbrace{\frac{1}{1+r}\tilde{E}_n\, g(S_{n+1})} \right\}$$

Just showed this is $\geq g(S_n)$

$\Rightarrow$ Holding the option longer always has a better expected return than exercising.

QED.

6.6. **Optimal Stopping.**

Definition 6.68. We say an adapted process M is a *super-martingale* if $\boldsymbol{E}_n M_{n+1} \leqslant M_n$.

Definition 6.69. We say an adapted process M is a *sub-martingale* if $\boldsymbol{E}_n M_{n+1} \geqslant M_n$.

Example 6.70. The discounted arbitrage free price of an American option is a *super-martingale* under the risk neutral measure.

Theorem 6.71 (Doob decomposition)**.** *Any adapted process can be uniquely expressed as the sum of a martingale and a predictable process that starts at* 0*. That is, if* X *is an adapted process there exists a unique pair of process* M, A *such that* M *is a martingale,* A *is predictable,* $A_0 = 0$ *and* $X = M + A$*.*

Proposition 6.72. *If X is a super-martingale, then there exists a unique martingale M and increasing predictable process A such that $X = M - A$.*

Proposition 6.73. *If X is a sub-martingale, then there exists a unique martingale M and increasing predictable process A such that $X = M + A$.*