Lecture 23 (10/25). Please enable your video if you can.

Last time: American option $\longrightarrow$ Intrinsic value G.

Exercise at any stopping time τ (your choice)

Collect intr value, G_τ

<u>Strategy I</u>: Have an american option.

Resell it as an option with fixed (<u>random</u>) maturity time $\underline{\tau}$ & payoff G_τ

AFP is V_0^τ

Price American option by selling to highest bidder.

i.e. $V_0 = \max_{\tau} V_0^{\tau}$ over all finite stopping times τ.

Let τ^* be a stopping time for which $V_0 = V_0^{\tau^*} = \max_{\tau} V_0^{\tau}$ (optimal exercise time).

Strategy II: Sell an American op to an investor for X_0 \$.

investor can cash out at any finite stopping time τ.

① Need to ensure my wealth $X_\tau \geq G_\tau \quad \forall \tau$.

(i.e. $\underline{\underline{X_n \geq G_n}} \quad \forall n$ a.s.)

② Also, for at least one stopping time $\underline{\underline{\tau^*}}$, need $\underline{X_{\tau^*} = G_{\tau^*}}$.

Question 6.57. *Does Strategy I replicate an American option? Say σ^* is the optimal exercise time, and we create a replicating portfolio (with wealth process X) for the option with payoff G_{σ^*} at time σ^*. Suppose an investor cashes out the American option at time τ. Can we pay him?*

(Strategy I doesn't immediately tell me how to invest X_0 so that we can replicate the American Option)

IOU: Use strat I to replicate.

Question 6.58. *Does Strategy II yield the same price as Strategy I? I.e. must* $X_0 = \max\{V_0^\sigma \mid \sigma \text{ is a finite stopping time }\}$*?*

Claim: Yes (Needs Proof IOU)

Question 6.59. *Is the wealth of the replicating portfolio (for an American option) uniquely determined?*

Replication:

Needed ① $X_n \geq G_n \quad \forall n.$

& ② For some τ^*, $X_{\tau^*} = G_{\tau^*}$

① $Y_n \geq G_n \quad \forall n.$

② $Y_{\tilde{\tau}^*} = G_{\tilde{\tau}^*}$ for some $\tilde{\tau}^*$

($\tilde{\tau}^*$ need not equal τ^*)

Must $X_n = Y_n$?

(Not immediately clear)

Question 6.60. *How do you find the minimal optimal exercise time, and the arbitrage free price? Let's take a simple example first.*

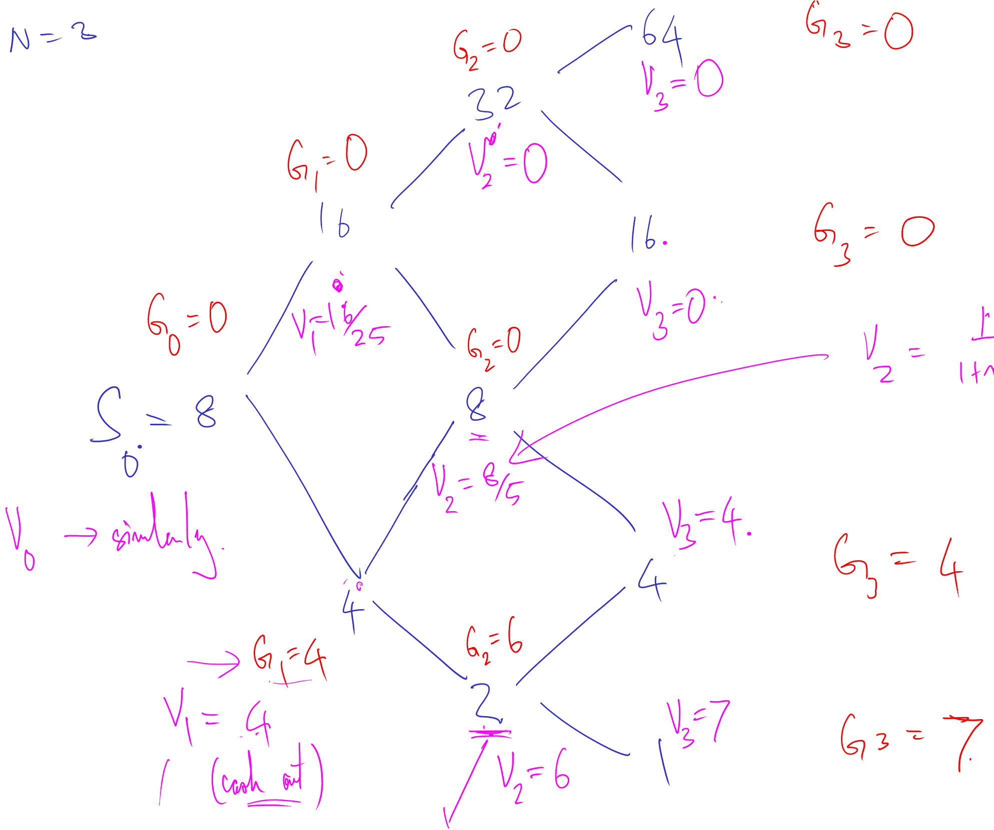
Eg: $u = 2$, $d = 1/2$, $r = 1/4$

$$\tilde{p} = \frac{1+r-d}{u-d} = \frac{5/4 - \frac{1}{2}}{3/2} = \frac{3/4}{3/2} = 1/2.$$

$$\tilde{q} = \qquad = 1/2.$$

~~Americke~~ American put strike $K = 8$. $(N = 3)$

$N = 3$



Pink = What I think the AFP should be.

$$V_2 = \frac{1}{1+r}\left(\tilde{p}\,0 + \tilde{q}\cdot 4\right) = \frac{4}{5}\left(\frac{4}{2}\right) = \frac{8}{5}$$

$$V_1(1,*) = \frac{4}{5}\left(\frac{1}{2}\,\frac{8}{5}\right) = \frac{16}{25}$$

Cash out $\longrightarrow 6$ (Better to cash out!)

Wait $\longrightarrow \frac{4}{5}\left(\frac{1}{2}\cdot 4 + \frac{1}{2} 7\right) = \frac{2(11)}{5} = \boxed{\frac{22}{5}}$

≈ 40 ...

Then Wait $\longrightarrow$ get $\frac{4}{5}\left(\frac{1}{2}\cdot\frac{8}{5} + \frac{6}{2}\right) = \frac{2}{5}\left(\frac{8}{5} + \frac{30}{5}\right) = \frac{76}{25}$

$\approx 3.0\ldots$ \$.

Theorem 6.61. *Consider the binomial model with $0 < d < 1 + r < u$, and an American option with intrinsic value G. Define*

$$V_N = G_N \,, \qquad V_n = \max\Big\{\frac{1}{D_n}\tilde{\boldsymbol{E}}_n(D_{n+1}V_{n+1}), G_n\Big\} \,, \qquad \sigma^* = \min\{n \leqslant N \mid V_n = G_n\} \,.$$

Then V_n is the arbitrage free price, and σ^* is the minimal optimal exercise time. Moreover, this option can be replicated.

Remark 6.62. The above is true in any complete, arbitrage free market.

Remark 6.63. In the Binomial model the above simplifies to:

$$V_n(\omega) = \max\Big\{\frac{1}{1+r}\Big(\tilde{p}V_{n+1}(\omega', 1) + \tilde{q}V_{n+1}(\omega', -1)\Big), G_n(\omega)\Big\} \,, \qquad \text{where } \omega = (\omega', \omega_{n+1}, \omega'') \,, \quad \omega' = (\omega_1, \dots, \omega_n) \,.$$

IOU a proof.