Lecture 22 (10/22). Please enable video if you can

Last time: Doob OST: $\tau \to$ bdd stopping time $\Bigg\}$ $E_n M_{(\tau)} = M_{n \wedge \tau}$

$M \to M_g$

(Note: For ind'g many coin tosses, OST is true if τ is bdd.

If $\tau < \infty$ a.s., then you typically need an extra cond.)

Consider a market with a few risky assets and a bank.

Proposition 6.49. *Suppose a market admits a risk neutral measure. If X is the wealth of a self-financing portfolio and τ is a bounded stopping time such that $X_0 = 0$, and $X_\tau \geqslant 0$, then $X_\tau = 0$. That is, there cant be an arbitrage opportunity at any bounded stopping time.*

(Under a RNM, X self fin $\iff$ $D_n X_n$ is a $\tilde{P}$ mg)

NTS $X_\tau = 0$ a.s. Know $D_n X_n$ is a $\tilde{P}$ mg. (self fin).

$\Rightarrow$ OST $\tilde{E}(D_\tau X_\tau) = \tilde{E}_0(D_\tau X_\tau) = D_0 X_0 = 0$

Know $D_\tau X_\tau \geqslant 0$ & $\tilde{E}(D_\tau X_\tau) = 0$

$\Rightarrow D_\tau X_\tau = 0$ a.s. $\Rightarrow X_\tau = 0$ a.s.

QED

Question 6.50 (Gamblers ruin). *Suppose* $N = \infty$. *Let* ξ_n *be i.i.d. random variables with mean* 0, *and let* $X_n = \sum_1^n \xi_k$. *Let* $\tau = \min\{n \mid X_n = 1\}$. *(It is known that* $\tau < \infty$ *almost surely.) What is* $\boldsymbol{E}X_\tau$*? What is* $\lim_{N\to\infty} \boldsymbol{E}X_{\tau\wedge N}$*?*

Note X is a Mg

$EX_\tau = 1 \quad (\neq 0)$

(Does not violate OST as τ need not be bdd)

Game: iid coin tosses

Win \$1 if heads

Lose \$1 if tails.

Strategy: Play until you win \$$10^6$ & leave.

$\tau =$ first time to reach \$ 10^6 (a stopping time)

Know $\tau < \infty$ a.s.

Q: $EX_{\tau\wedge N} \overset{OST}{=} E_0 X_{\tau \wedge N} = X_0 = 0$

0 ($\tau \wedge N$ is a bdd stopping time)

$0 = \lim_{N\to\infty} EX_{\tau\wedge N} \neq EX_\tau = 1$

(1) The Game is fair!

(2) What is $E X_\tau$? $\quad (X_n = \text{wealth at time } n)$.

$E X_\tau = 10^6$ (Does not contradict OST: τ need not be bdd).

$\boxed{\text{(3) Have we beaten the House?}}$

6.5. **American Options.** An American option is an option that can be exercised at any time chosen by the holder.

Definition 6.51. Let $G_0, G_1, \ldots, G_N$ be an adapted process. An *American option* with *intrinsic value* G is a security that pays G_σ at any finite stopping time σ chosen by the holder.

Example 6.52. An *American put* with strike K is an American option with intrinsic value $(K - S_n)^+$.

Question 6.53. *How do we price an American option? How do we decide when to exercise it? What does it mean to replicate it?*

Strategy I: Let σ be a finite stopping time, and consider an option with (random) maturity time σ and payoff G_σ. Let V_0^σ denote the arbitrage free price of this option. The arbitrage free price of the American option *should be* $V_0 = \max_\sigma V_0^\sigma$, where the maximum is taken over all *finite* stopping times σ.

Definition 6.54. The *optimal exercise time* is a stopping time σ^* that maximizes $V_0^{\sigma^*}$ over all finite stopping times.

Definition 6.55. An optimal exercise time σ^* is called *minimal* if for every optimal exercise time τ^* we have $\sigma^* \leqslant \tau^*$.

Remark 6.56. The optimal exercise time need not be unique. (The *minimal* optimal exercise time is certainly unique.)

$V_0^\sigma \to$ AFP of a option that matures at σ & pays G_σ.

Know American option is worth more than any of these options.

Guess: AFP of American opt $= V_0 = \max_\sigma V_0^\sigma$

$(V_0 = V_0^{\sigma^*})$.

Question 6.57. *Does this replicate an American option? Say σ^* is the optimal exercise time, and we create a replicating portfolio (with wealth process X) for the option with payoff G_{σ^*} at time σ^*. Suppose an investor cashes out the American option at time τ. Can we pay him?*

Strategy II: Replication. Suppose we have sold an American option with intrinsic value G to an investor. Using that, we hedge our position by investing in the market/bank, and let X_n be the our wealth at time n.

(1) Need $X_\sigma \geqslant G_\sigma$ for all finite stopping times σ. (Or equivalently $X_n \geqslant G_n$ for all n.)
(2) For (at-least) *one* stopping time σ^*, need $X_{\sigma^*} = G_{\sigma^*}$.

The arbitrage free price of this option is X_0.

Sell American opt for X_0 at time 0

Invest X_0 $\longrightarrow$ Wealth X_n at time n.