Lecture 21 (10/20). Please enable video if you can

Last time: Up rebate option: pays face value A the first time stock price exceeds U

Found the AFP as $\mathbb{1}_{\{n \le \tau\}} V_n = \mathbb{1}_{\{n \le \tau\}} f_n(S_n)$

τ = time option pays / expires $= \left(\min\{k \le N \mid S_k \ge U\}\right) \wedge N$

& found a recurrence relation for f_n in terms of f_{n+1}

Proposition 6.44. *Let $Y = (Y^1, \dots, Y^d)$ be a d-dimensional process such that for every n we have $Y_{n+1}(\omega) = h_{n+1}(Y_n(\omega), \omega_{n+1})$ for some deterministic function h_{n+1}. Let $A_1, \dots, A_N \subseteq \mathbb{R}^d$, with $A_N \subseteq \mathbb{R}^d$, and define the stopping time σ by*

$$\sigma = \min\{n \in \{0, \dots, N\} \mid Y_n \in A_n\} .$$

Let $g_0, \dots g_N$ be N deterministic functions on $\mathbb{R}^d$, and consider a security that pays $G_\sigma = g_\sigma(Y_\sigma)$. The arbitrage free price of this security is of the form $V_n \mathbf{1}_{\{\sigma \geqslant n\}} = f_n(Y_n)\mathbf{1}_{\{\sigma \geqslant n\}}$. The functions f_n satisfy the recurrence relation

$$f_N(y) = g_N(y)$$

$$f_n(y) = \mathbf{1}_{\{y \in A_n\}} g_n(y) + \frac{\mathbf{1}_{\{y \notin A_n\}}}{1+r} \Big(\tilde{p} f_{n+1}(h_{n+1}(y, 1)) + \tilde{q} f_{n+1}(h_{n+1}(y, -1)) \Big)$$

Y^1 (superscript) $\longrightarrow$ coordinate

Y_1 (sub script) $\longrightarrow$ time.

For up rebate option.

$A_1 = [U, \infty)$

$d = 1$

$Y^1 = S^{u}$

6.4. **Optional Sampling.**

Theorem 6.45 (Doob's optional sampling theorem). *Let τ be a bounded stopping time and M be a martingale. Then $\boldsymbol{E}_n M_\tau = M_{\tau \wedge n}$.*

Remark 6.46. When dealing with finitely many coin tosses ($N < \infty$), *bounded stopping times* are the same as *finite stopping times*. When dealing with infinitely many coin tosses, the two notions are different.

Remark 6.47. When $N = \infty$ and τ is not bounded, the optional sampling theorem is still true if $X_{\tau \wedge k}$ is uniformly bounded in k.

Corollary 6.48. *If M is a martingale and τ is a bounded stopping time, then $\boldsymbol{E} M_\tau = \boldsymbol{E} M_0$.*

Note : Fix $\tau = n+1$ (is a stopping time)

$$E_n M_\tau = E_n M_{n+1} = M_n \text{ (def of Mg)}$$

$\|$ OST

$$M_n = M_{(n+1)\wedge n} = M_{\tau \wedge n}$$

Say M is a mg $\longrightarrow$ fair game

$\Rightarrow EM_n = EM_0 = M_0$ ($\because M_0$ is not random).

$\hookrightarrow$ Q: Say we stop playing at some bdd stopping time $\underline{\underline{\tau}}$

then what is EM_τ ? <u>Claim</u>: $EM_\tau = EM_0$

Proof: By OST: $\overset{(n=0)}{E_0} M_{\tau \wedge 0} \overset{OST}{=} M_0$

$= EM_0$ ($\because M_0$ is not random)

Proof of Theorem 6.45 τ is a bdd stopping time

NTS $E_n M_\tau = M_{\tau \wedge n}$

Note $E_n M_\tau = E_n \left(\sum_{k=0}^{n} \underbrace{1_{\{\tau=k\}}}_{\mathcal{F}_n\text{-meas}} M_{\underset{k}{\tau}} + \sum_{k=n+1}^{N} 1_{\{\tau=k\}} M_\tau \right)$

$$= E_n \left(\underbrace{\sum_{k=0}^{n} 1_{\{\tau=k\}} M_k}_{\mathcal{F}_n\text{-meas.}} \right) + E_n \left(\sum_{k=n+1}^{N} 1_{\{\tau=k\}} M_\tau \right)$$

ℓ

$$= \sum_{k=0}^{n} \mathbb{1}_{\{\tau=k\}} M_{n\wedge\tau} \quad + \quad \|$$

$$= \mathbb{1}_{\{\tau\le n\}} M_{\tau\wedge n} \quad + \quad E_n\left(\sum_{k=n+1}^{N} \mathbb{1}_{\{\tau=k\}} M_{\tau}\right)$$

$$= \quad \| \quad + \quad \sum_{k=n+1}^{N} E_n\left(\underbrace{\mathbb{1}_{\{\tau=k\}} M_k}_{\mathcal{F}_k\text{-meas}\ (k\ge n)}\right)$$

$$= \quad \| \quad + \sum_{k=n+1}^{N} E_n E_k \left(\mathbb{1}_{\{\tau=k\}} M_k \right)$$

$$= \quad \| \quad + \sum_{k=n+1}^{N} E_n \left(\mathbb{1}_{\{\tau=k\}} \underline{\underline{E_k M_k}} \right)$$

$$= \quad \| \quad + \sum_{k=n+1}^{N} E_n \left(\mathbb{1}_{\{\tau=k\}} E_k M_N \right) \quad \left(\because M \text{ is a mg} \right)$$

$$= \text{''} + \sum_{k=n+1}^{N} E_n \left(E_k \left(\mathbb{1}_{\{\tau=k\}} M_N \right) \right)$$

$$= \text{''} + \sum_{k=n+1}^{N} E_n \left(\mathbb{1}_{\{\tau=k\}} M_N \right) \qquad (\text{tower prop.})$$

$$= \text{''} + E_n \left[\underbrace{\left(\sum_{k=n+1}^{N} \mathbb{1}_{\{\tau=k\}} \right)}_{} M_N \right]$$

$$= \quad '' \quad + E_n\left(\underbrace{\mathbb{1}_{\{\tau > n\}}}_{\mathcal{F}_n\text{-meas.}} M_N \right)$$

$$= \quad '' \quad + \mathbb{1}_{\{\tau > n\}} E_n M_N$$

$$= \quad '' \quad + \mathbb{1}_{\{\tau > n\}} M_n =$$

$$= \mathbb{1}_{\{\tau \le n\}} M_{\tau \wedge n} + \mathbb{1}_{\{\tau > n\}} M_{\tau \wedge n} = M_{\tau \wedge n} \quad \text{QED.}$$