Lecture 20 (10/18). Please enable your video if you can.

Last

- Let G be an adapted process, and σ be a *finite* stopping time.
- Note $G_\sigma = \sum_{n=}^{N} G_n \mathbf{1}_{\sigma=n}$.
- Let $(X_0, (\Delta_n))$ be a self-financing portfolio, and X_n at time n be the wealth of this portfolio at time n.

Last time.

Definition 6.38. Consider a derivative security that pays G_σ at the random time σ. A self-financing portfolio with wealth process X is a replicating strategy if $X_\sigma = G_\sigma$.

Remark 6.39. If a replicating strategy exists, then at any time before σ, the wealth of the replicating strategy must equal the arbitrage free price V. That is, $\mathbf{1}_{\{n \leqslant \sigma\}} X_n = \mathbf{1}_{\{n \leqslant \sigma\}} V_n$.

Theorem 6.40. *The security with payoff G_σ (at the stopping time σ) can be replicated. The arbitrage free price is given by*

$$\boxed{V_n \mathbf{1}_{\{\sigma \geqslant n\}} = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n (D_\sigma G_\sigma \mathbf{1}_{\{\sigma \geqslant n\}})}.$$

Remark 6.41. The only thing required for the proof of Theorem 6.40 is the fact that X_n is the wealth of a self-financing portfolio if and only if $D_n X_n$ is a $\tilde{\boldsymbol{P}}$ martingale.

$$\mathbb{1}_{\{n \leq \sigma\}} V_n = \mathbb{1}_{\{n \leq \sigma\}} \frac{1}{D_n} \tilde{E}_n (D_\sigma G_\sigma)$$

Proposition 6.42. *The wealth of the replicating portfolio (at times before σ) is uniquely determined by the recurrence relations:*

$$X_N \mathbf{1}_{\{\sigma=N\}} = G_N \mathbf{1}_{\{\sigma=N\}}$$

$$X_n \mathbf{1}_{\{\sigma\geqslant n\}} = G_n \mathbf{1}_{\{\sigma=n\}} + \frac{1}{1+r} \mathbf{1}_{\{\sigma>n\}} \tilde{\boldsymbol{E}}_n X_{n+1} .$$

If we write $\omega = (\omega', \omega_{n+1}, \omega'')$ with $\omega' = (\omega_1, \dots, \omega_n)$, then we know in the Binomial model we have

$$\tilde{\boldsymbol{E}}_n X_{n+1}(\omega) = \tilde{\boldsymbol{E}}_n X_{n+1}(\omega') = \tilde{p} X_{n+1}(\omega', 1) + \tilde{q} X_{n+1}(\omega', -1) .$$

(Recall X_n = wealth of rep port = AFP = $\underline{V_n}$)

Pf: Rep port is self fin $\Rightarrow D_n X_n$ is a $\tilde{P}$ mg

$\Rightarrow D_n X_n = \tilde{E}_n (D_{n+1} X_{n+1}) \Rightarrow X_n = \frac{1}{D_n} \tilde{E}_n (D_{n+1} X_{n+1})$

$= \frac{1}{1+r} \tilde{E}_n X_{n+1}$

$$\Rightarrow \mathbb{1}_{\{\tau \geq n\}} X_n = \underbrace{\mathbb{1}_{\{\tau \geq n\}}}_{\mathcal{F}_n\text{- meas}} \frac{1}{1+r} \tilde{E}_n X_{n+1}$$

$$= \frac{1}{1+r} \left(\tilde{E}_n \left(\mathbb{1}_{\{\tau = n\}} X_{n+1} + \mathbb{1}_{\{\tau > n\}} X_{n+1} \right) \right)$$

$$= \mathbb{1}_{\{\tau = n\}} \tilde{E}_n \frac{X_{n+1}}{1+r} + \frac{1}{1+r} \mathbb{1}_{\{\tau > n\}} \tilde{E}_n X_{n+1}$$

$$= \frac{1}{D_n} \mathbb{1}_{\{\tau = n\}} \tilde{E}_n \left(D_{n+1} X_{n+1} \right) + \frac{1}{1+r} \mathbb{1}_{\{\tau > n\}} \tilde{E}_n X_{n+1}$$

$$= \frac{1}{\cancel{D_n}} \mathbb{1}_{\{\sigma = n\}} (\cancel{D_n} X_n) +$$

$$\|$$

$$n$$

$$= \mathbb{1}_{\{\sigma = n\}} X_\sigma +$$

$$= \mathbb{1}_{\{\sigma = n\}} G_\sigma + \frac{1}{1+r} \mathbb{1}_{\{\sigma > n\}} E_n X_{n+1}$$

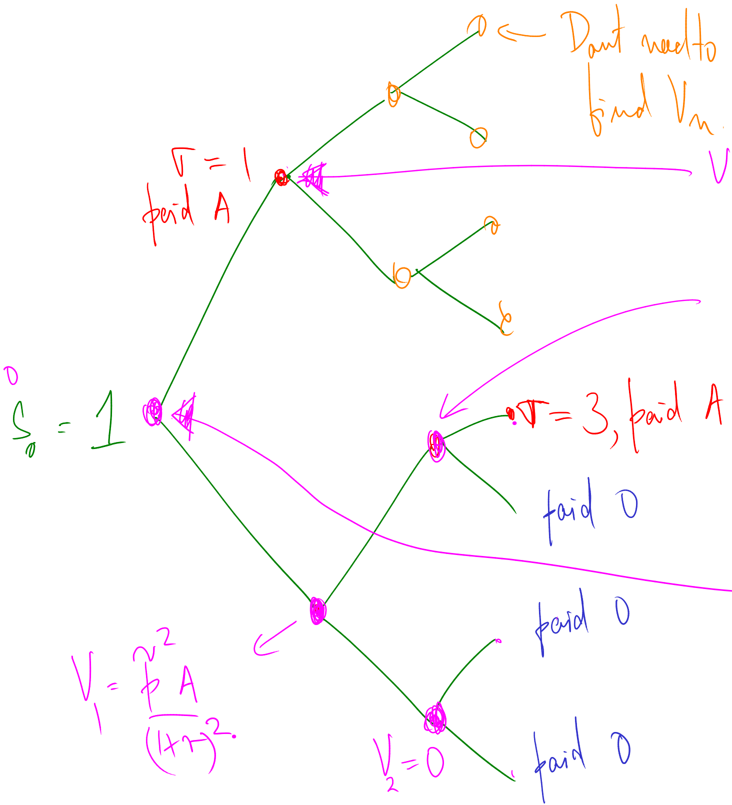
QED.

As before, we will use state processes to find practical algorithms to price securities.

Example 6.43. Let $A, U > 0$. The up-and-rebate option pays the face value A at the first time the stock price exceeds U (up to maturity time N), and nothing otherwise. Find an efficient way to compute the arbitrage free price of this option.

Say $d = \frac{1}{u}$, $S_0 = 1$, $U = uS_0 = u$, $N = 3$.

Option 1: Draw tree of all coin tosses up to $N = 3$ & price.



V_u

$V_1 = A$

$$V_2 = \frac{\tilde{p} A}{1+r} + 0\tilde{q}$$

$$V_0 = \frac{1}{1+r}\left(\tilde{p} A + \tilde{q}\,\frac{\tilde{p}^2 A}{(1+r)^2}\right)$$

$$V_1 = \frac{\tilde{p}^2 A}{(1+r)^2}.$$

$V_2 = 0$

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Try same example: But look for $\mathbb{1}_{\{n \le \sigma\}} V_n = \mathbb{1}_{\{n \le \sigma\}} f_n(S_n)$.

(1) $\mathbb{1}_{\{\sigma = N\}} V_N = \begin{cases} A & \text{if } S_N \geq u \\ 0 & \text{if not} \end{cases}$.

(Note $V_N \neq \begin{cases} A & \text{if } S_N \geq u \\ 0 & \text{o/w} \end{cases}$, but $\mathbb{1}_{\{\sigma = N\}} V_N = \begin{cases} A & \text{if } S_N \geq u \\ 0 & \text{o/w} \end{cases}$)

u

② Say $\mathbb{1}_{\{\sigma \geq n+1\}} V_{n+1} = \mathbb{1}_{\{\sigma \geq n+1\}} f_{n+1}(S_{n+1})$ $\quad (n \leq N-1)$

$\left(\text{Know } f_N(s) = A\, \mathbb{1}_{\{s \geq U\}} \right)$

See if we can find a fn f_n $\ni$ $\mathbb{1}_{\{\sigma \geq n\}} V_n = \mathbb{1}_{\{\sigma \geq n\}} f_n(S_n)$

Know $\mathbb{1}_{\{\sigma \geq n\}} V_n = \overbrace{\mathbb{1}_{\{\sigma = n\}} A} + \mathbb{1}_{\{\sigma > n\}} \frac{1}{1+r} \tilde{E}_n V_{n+1}$

$$= \mathbb{1}_{\{\tau \geq n\}} \left(A \, \mathbb{1}_{\{S_n \geq U\}} + \mathbb{1}_{\{S_n < U\}} \frac{1}{1+r} \tilde{E}_n f_{n+1}(S_{n+1}) \right)$$

$$= \mathbb{1}_{\{\tau \geq n\}} \left(A \, \mathbb{1}_{\{\underline{S_n} \geq U\}} + \mathbb{1}_{\{\underline{S_n} < U\}} \left(\frac{f_{n+1}(u\underline{S_n}) \, \tilde{p} + f_{n+1}(d\underline{S_n}) \, \tilde{q}}{1+r} \right) \right)$$

$$= \mathbb{1}_{\{\tau \geq n\}} \cdot \underline{f_n}(S_n), \quad \text{where}$$

recurrence relation!

$$\boxed{f_n(s) = A \, \mathbb{1}_{\{s \geq U\}} + \mathbb{1}_{\{s < U\}} \left(\frac{f_{n+1}(us)\,\tilde{p} + f_{n+1}(ds)\,\tilde{q}}{1+r} \right)}$$

Proposition 6.44. *Let $Y = (Y^1, \ldots, Y^d)$ be a d-dimensional process such that for every n we have $Y_{n+1}(\omega) = h_{n+1}(Y_n(\omega), \omega_{n+1})$ for some deterministic function h_{n+1}. Let A_1, ..., $A_N \subseteq \mathbb{R}^d$, with A_N $\mathbb{R}^d$, and define the stopping time σ by*

$$\sigma = \min\{n \in \{0, \ldots, N\} \mid Y_n \in A_n\}\,.$$

Let g_0, $\ldots g_N$ be N deterministic functions on $\mathbb{R}^d$, and consider a security that pays $G_\sigma = g_\sigma(Y_\sigma)$. The arbitrage free price of this security is of the form $V_n \mathbf{1}_{\{\sigma \geqslant n\}} = f_n(Y_n)\mathbf{1}_{\{\sigma \geqslant n\}}$. The functions f_n satisfy the recurrence relation

$$f_N(y) = g_N(y)$$

$$f_n(y) = \mathbf{1}_{\{y \in A_n\}} g_n(y) + \frac{\mathbf{1}_{\{y \notin A_n\}}}{1+r}\Big(\tilde{p} f_{n+1}(h_{n+1}(y,1)) + \tilde{q} f_{n+1}(h_{n+1}(y,-1))\Big)$$