

Lecture 19 (10/15). Please enable your video if you can.

$$a(\underline{bc}) = (\underline{ab})c \quad (\text{but not on a computer!})$$

$$\hookrightarrow \text{error} \approx 10^{-17}$$

$$\text{range} \cdot (S_{n+1}) \xrightarrow{\text{on computer}} \{us \mid s \in \text{Range}(S_n)\} \cup \{dc \mid s \in \text{Range}(S_n)\}.$$

lots of repeats

smaller range $\rightarrow$ less computations.

last time:

or stopping time: play a game. Stop at time τ . (random)

Need $\{\tau = n\} \in \mathcal{F}_n$ (only dep on 1st n coin tosses)

↑
stop at time n

Def: τ is a stopping time if (1) $\tau: \Omega \rightarrow \{0, 1, \dots, n\} \cup \{\infty\}$

& (2) Need $\{\tau = n\} \in \mathcal{F}_n \quad \forall n$.

(Note: (2) $\Leftrightarrow$ (2'): Need $\{\tau \leq n\} \in \mathcal{F}_n \quad \forall n$).

- Let $\underline{G}$ be an adapted process, and $\underline{\sigma}$ be a *finite* stopping time.
- Note $\underline{G}_{\underline{\sigma}} = \sum_{n=0}^N \underline{G}_n \mathbf{1}_{\underline{\sigma} \geq n}$.
- Let $(\underline{X}_0, (\underline{\Delta}_n))$ be a self-financing portfolio, and $\underline{X}_n$ at time n be the wealth of this portfolio at time n .

Definition 6.38. Consider a derivative security that pays $\underline{G}_{\underline{\sigma}}$ at the random time $\underline{\sigma}$. A self-financing portfolio with wealth process $\underline{X}$ is a replicating strategy if $\underline{X}_{\underline{\sigma}} = \underline{G}_{\underline{\sigma}}$.

Remark 6.39. If a replicating strategy exists, then at any time before $\underline{\sigma}$, the wealth of the replicating strategy must equal the arbitrage free price V . That is, $\mathbf{1}_{\{n \leq \underline{\sigma}\}} \underline{X}_n = \mathbf{1}_{\{n \leq \underline{\sigma}\}} \underline{V}_n$.

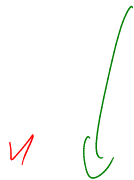
Theorem 6.40. The security with payoff $\underline{G}_{\underline{\sigma}}$ (at the stopping time $\underline{\sigma}$) can be replicated. The arbitrage free price is given by

$$\underline{V}_n \mathbf{1}_{\{\underline{\sigma} \geq n\}} = \frac{1}{D_n} \tilde{\mathbb{E}}_n(D_{\underline{\sigma}} \underline{G}_{\underline{\sigma}} \mathbf{1}_{\{\underline{\sigma} \geq n\}})$$

Remark 6.41. The only thing required for the proof of Theorem 6.40 is the fact that $\underline{X}_n$ is the wealth of a self-financing portfolio if and only if $D_n \underline{X}_n$ is a $\tilde{\mathbf{P}}$ martingale.

Note $\{\underline{\sigma} \geq n\} \in \mathcal{F}_n \Rightarrow \frac{1}{\mathbf{1}_{\{\underline{\sigma} \geq n\}}}$ is $\mathcal{F}_n$ - meas

$$\Rightarrow \tilde{\mathbb{E}}_n \left(\frac{1}{\mathbf{1}_{\{\underline{\sigma} \geq n\}}} D_{\underline{\sigma}} \underline{G}_{\underline{\sigma}} \right) = \frac{1}{\mathbf{1}_{\{\underline{\sigma} \geq n\}}} \tilde{\mathbb{E}}_n (D_{\underline{\sigma}} \underline{G}_{\underline{\sigma}})$$



Note $\{\tau = n\} \in \mathcal{F}_n$. ($\because \tau$ is a stopping time)

$$\{\tau \leq n\} \in \mathcal{F}_n \quad (\text{Yes: } \{\tau \leq n\} = \bigcup_{k=0}^n \underbrace{\{\tau = k\}}_{\in \mathcal{F}_k} \in \mathcal{F}_n)$$

$$\{\tau > n\} = \underbrace{\{\tau \leq n\}^c}_{\in \mathcal{F}_n}$$

Also $\{\tau > n\} \in \mathcal{F}_n$.

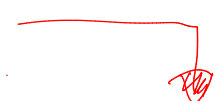
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Pr of Thm 6.40: let $X_n = \frac{1}{D_n} \tilde{E}_n(D_\sigma G_\sigma)$

Claim (1) $X = \text{wealth of a self fin portfolio}$ } $\Rightarrow X$ is a rep portfolio.
 Claim (2) $X_T = G_\sigma$

Once we know $X = \text{wealth of rep port}$,

$$\begin{aligned} \text{Then } V_n \mathbb{1}_{\{n \leq \sigma\}} &= X_n \mathbb{1}_{\{n \leq \sigma\}} = \mathbb{1}_{\{n \leq \sigma\}} \frac{1}{D_n} \tilde{E}_n(D_\sigma G_\sigma) \\ &= \frac{1}{D_n} \tilde{E}_n(D_\sigma G_\sigma \mathbb{1}_{\{n \leq \sigma\}}) \end{aligned}$$

f_n - meas 

= formula we wanted to prove.

Pf of claim 1: X is wealth of a self fin part

$\Leftrightarrow D_n X_n$ is a $\tilde{P}$ mg

Only NTS $D_n X_n$ is a $\tilde{P}$ mg.

compute
$$\tilde{E}_n(D_{n+1} X_{n+1}) = \tilde{E}_n \cancel{D_{n+1}} \tilde{E}_{n+1} \overbrace{(D_\sigma G_\sigma)}^{\text{formula for } X_{n+1}} \cdot \cancel{\frac{1}{D_{n+1}}}$$

therefore
$$\tilde{E}_n(D_\sigma G_\sigma) = D_n X_n \quad (\text{formula for } X_n).$$

QED (Claim 1).

Pf of claim ②: NTS $X_\sigma = G_\sigma$.

Enough to show $\forall n, \quad \mathbb{1}_{\{\sigma=n\}} X_\sigma = \mathbb{1}_{\{\sigma=n\}} G_\sigma$.

lets prove this: LHS = $\mathbb{1}_{\{\sigma=n\}} X_{\underline{\sigma}} = \mathbb{1}_{\{\sigma=n\}} X_{\underline{n}}$

$$= \mathbb{1}_{\{\sigma=n\}} \frac{1}{D_n} \tilde{E}_n(D_\sigma G_\sigma)$$

$$\underbrace{F_n}_{F_n - \text{meas}} \rightarrow$$

$$= \frac{1}{D_n} \tilde{E}_n \left(\mathbb{1}_{\{\tau=n\}} \underbrace{D_\sigma}_{\text{red}} \underbrace{G_\sigma}_{\text{red}} \right) = \frac{1}{D_n} \tilde{E}_n \left(\mathbb{1}_{\{\sigma=n\}} \underbrace{D_n}_{\text{red}} \underbrace{G_n}_{\text{red}} \right)$$

$$\underbrace{\hspace{10em}}_{F_n \text{ meas}}$$

$$= \frac{1}{\cancel{D_n}} \mathbb{1}_{\{\sigma=n\}} \cancel{D_n} G_n = \mathbb{1}_{\{\tau=n\}} G_n$$

QED,

Proposition 6.42. *The wealth of the replicating portfolio (at times before σ) is uniquely determined by the recurrence relations:*

$$\left. \begin{aligned} X_N \mathbf{1}_{\{\sigma=N\}} &= G_N \mathbf{1}_{\{\sigma=N\}} \\ X_n \mathbf{1}_{\{\sigma \geq n\}} &= G_n \mathbf{1}_{\{\sigma=n\}} + \frac{1}{1+r} \mathbf{1}_{\{\sigma > n\}} \tilde{E}_n X_{n+1} \end{aligned} \right\}$$

If we write $\omega = (\omega', \omega_{n+1}, \omega'')$ with $\omega' = (\omega_1, \dots, \omega_n)$, then we know in the Binomial model we have

$$\tilde{E}_n X_{n+1}(\omega) = \tilde{E}_n X_{n+1}(\omega') = \tilde{p} X_{n+1}(\omega', 1) + \tilde{q} X_{n+1}(\omega', -1).$$

As before, we will use state processes to find practical algorithms to price securities.

Example 6.43. Let $A, U > 0$. The up-and-rebate option pays the face value A at the first time the stock price exceeds U (up to maturity time N), and nothing otherwise. Find an efficient way to compute the arbitrage free price of this option.

Proposition 6.44. *Let $Y = (Y^1, \dots, Y^d)$ be a d -dimensional process such that for every n we have $Y_{n+1}(\omega) = h_{n+1}(Y_n(\omega), \omega_{n+1})$ for some deterministic function h_{n+1} . Let $A_1, \dots, A_N \subseteq \mathbb{R}^d$, with $A_N \neq \emptyset$, and define the stopping time σ by*

$$\sigma = \min\{n \in \{0, \dots, N\} \mid Y_n \in A_n\}.$$

Let $g_0, \dots, g_N$ be N deterministic functions on $\mathbb{R}^d$, and consider a security that pays $G_\sigma = g_\sigma(Y_\sigma)$. The arbitrage free price of this security is of the form $V_n \mathbf{1}_{\{\sigma \geq n\}} = f_n(Y_n) \mathbf{1}_{\{\sigma \geq n\}}$. The functions f_n satisfy the recurrence relation

$$f_N(y) = g_N(y)$$

$$f_n(y) = \mathbf{1}_{\{y \in A_n\}} g_n(y) + \frac{\mathbf{1}_{\{y \notin A_n\}}}{1+r} \left(\tilde{p} f_{n+1}(h_{n+1}(y, 1)) + \tilde{q} f_{n+1}(h_{n+1}(y, -1)) \right)$$

6.4. Optional Sampling. Consider a market with a few risky assets and a bank.

Question 6.45. *If there is no arbitrage opportunity at time N , can there be arbitrage opportunities at time $n \leq N$? How about at finite stopping times?*

Proposition 6.46. *There is no arbitrage opportunity at time N if and only if there is no arbitrage opportunity at any finite stopping time.*

Question 6.47. *Say M is a martingale. We know $EM_n = EM_0$ for all n . Is this also true for stopping times?*

Theorem 6.48 (Doob's optional sampling theorem). *Let τ be a bounded stopping time and M be a martingale. Then $\mathbf{E}_n M_\tau = M_{\tau \wedge n}$.*

Proposition 6.49. *Suppose a market admits a risk neutral measure. If X is the wealth of a self-financing portfolio and τ is a finite stopping time such that $X_0 = 0$, and $X_\tau \geq 0$, then $X_\tau = 0$.*

Remark 6.50. This is simply an alternate proof of Proposition [6.46](#).

Question 6.51 (Gamblers ruin). Suppose $N = \infty$. Let X_n be i.i.d. random variables with mean 0, and let $S_n = \sum_1^n X_k$. Let $\tau = \min\{n \mid S_n = 1\}$. (It is known that $\tau < \infty$ almost surely.) What is $\mathbf{E}S_\tau$? What is $\lim_{N \rightarrow \infty} \mathbf{E}S_{\tau \wedge N}$?

6.5. American Options. An American option is an option that can be exercised at any time chosen by the holder.

Definition 6.52. Let $G_0, G_1, \dots, G_N$ be an adapted process. An *American option* with *intrinsic value* G is a security that pays G_σ at any finite stopping time σ chosen by the holder.

Example 6.53. An *American put* with strike K is an American option with intrinsic value $(K - S_n)^+$.

Question 6.54. *How do we price an American option? How do we decide when to exercise it? What does it mean to replicate it?*