Lecture 19 (10/15). Please enable your video if you can.

$a(bc) = (ab)c$ (but not on a computer!)

↳ error $\approx 10^{-17}$

range: (S_{n+1}) $\xrightarrow{\text{on computer}}$ $\{us \mid s \in \mathrm{Range}(S_n)\} \cup \{ds \mid s \in \mathrm{Range}(S_n)\}$.. lots of repeats

Smaller range ⟶ less computations.

Last time:

Stopping time: Play a game. Stop at time τ. (random)

Need $\{\tau = n\} \in \mathcal{F}_n$ (only dep on 1^{st} n coin tosses)

stop at time n

Def: τ is a stopping time if ① $\tau : \Omega \longrightarrow \{0, 1, \cdots N\} \cup \{\infty\}$

& ② Need $\{\tau = n\} \in \mathcal{F}_n$ $\forall n$.

(Note: ② $\Longleftrightarrow$ ②': Need $\{\tau \leq n\} \in \mathcal{F}_n$ $\forall n$).

- Let G be an adapted process, and σ be a *finite* stopping time.
- Note $G_\sigma = \sum_{n=0}^{N} G_n \mathbf{1}_{\sigma=n}$.
- Let $(X_0, (\Delta_n))$ be a self-financing portfolio, and X_n at time n be the wealth of this portfolio at time n.

Definition 6.38. Consider a derivative security that pays G_σ at the random time σ. A self-financing portfolio with wealth process X is a replicating strategy if $X_\sigma = G_\sigma$.

Remark 6.39. If a replicating strategy exists, then at any time before σ, the wealth of the replicating strategy must equal the arbitrage free price V. That is, $\mathbf{1}_{\{n\leqslant\sigma\}} X_n = \mathbf{1}_{\{n\leqslant\sigma\}} V_n$.

Binomial model & $0 < d < 1+r < u$

Theorem 6.40. *The security with payoff G_σ (at the stopping time σ) can be replicated. The arbitrage free price is given by*

$$\boxed{V_n \mathbf{1}_{\{\sigma\geqslant n\}} = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n(D_\sigma G_\sigma \mathbf{1}_{\{\sigma\geqslant n\}})}$$

Remark 6.41. The only thing required for the proof of Theorem 6.40 is the fact that X_n is the wealth of a self-financing portfolio if and only if $D_n X_n$ is a $\tilde{\boldsymbol{P}}$ martingale.

Note $\{\sigma \geq n\} \in \mathcal{F}_n \Rightarrow \mathbb{1}_{\{\sigma \geq n\}}$ is $\mathcal{F}_n$ - meas

$$\Rightarrow \tilde{E}_n\left(\mathbb{1}_{\{\sigma \geq n\}} D_\sigma G_\sigma\right) = \mathbb{1}_{\{\sigma \geq n\}} \tilde{E}_n\left(D_\sigma G_\sigma\right)$$

Note $\{\tau = n\} \in \mathcal{F}_n$. ($\because$ τ is a stopping time)

$\{\tau \leq n\} \in \mathcal{F}_n$ (Yes: $\{\tau \leq n\} = \bigcup_{k=0}^{n} \underset{\mathcal{F}_k}{\underset{\cap}{\{\tau = k\}}} \in \mathcal{F}_n$.

$$\{\tau > n\} = \underbrace{\{\tau \leq n\}^c}_{\in \mathcal{F}_n}$$

Also $\{\tau > n\} \in \mathcal{F}_n$

n

Pf of Thm 6.40: Let $X_n = \frac{1}{D_n} \tilde{E}_n (D_\sigma G_\sigma)$

Claim ① X = wealth of a self fin fin portfolio $\Big\}\Rightarrow X$ is a rep portfolio.

Claim ② $X_\sigma = G_\sigma$

Once we know X = wealth of rep port,

$\mathcal{F}_n$-meas

Then $V_n \mathbb{1}_{\{n \le \sigma\}} = X_n \mathbb{1}_{\{n\le\sigma\}} = \mathbb{1}_{\{n\le\sigma\}} \frac{1}{D_n} \tilde{E}_n (D_\sigma G_\sigma)$

$= \frac{1}{D_n} \tilde{E}_n \left(D_\sigma G_\sigma \mathbb{1}_{\{n\le\sigma\}} \right)$

= formula we wanted to prove.

Pf of claim 1: X is wealth of a self fin port

$\iff D_n X_n$ is a $\tilde{P}$ mg

Only NTS $D_n X_n$ is a $\tilde{P}$ mg.

Compute $\tilde{E}_n(D_{n+1} X_{n+1}) = \tilde{E}_n \cancel{D_{n+1}} \underbrace{\tilde{E}_{n+1}(D_\sigma G_\sigma) \cdot \frac{1}{\cancel{D_{n+1}}}}_{\text{formula for } X_{n+1}}$

$\overset{\text{tower}}{=\!=} \tilde{E}_n(D_\sigma G_\sigma) = D_n X_n$ (formula for X_n).

QED (Claim 1).

Pf of claim 2: NTS $X_\tau = G_\tau$.

Enough to show $\forall n$, $\mathbb{1}_{\{\tau = n\}} X_\tau = \mathbb{1}_{\{\tau = n\}} G_\tau$.

Let's prove this: $LHS = \mathbb{1}_{\{\tau = n\}} X_{\underline{\underline{\tau}}} = \mathbb{1}_{\{\tau = n\}} X_{\underline{\underline{n}}}$

$$= \mathbb{1}_{\{\tau = n\}} \frac{1}{D_n} \tilde{E}_n (D_\tau G_\tau)$$

$\mathcal{F}_n$- meas

$$= \frac{1}{D_n} \tilde{E}_n\left(1_{\{\tau = n\}} D_\tau G_\tau \right) = \frac{1}{D_n} \tilde{E}_n\left(\underbrace{1_{\{\tau = n\}} D_n G_n}_{\mathcal{F}_n \text{ meas}} \right)$$

$$= \frac{1}{\cancel{D_n}} 1_{\{\tau = n\}} \cancel{D_n} G_n = 1_{\{\tau = n\}} G_n$$

QED.

Proposition 6.42. *The wealth of the replicating portfolio (at times before σ) is uniquely determined by the recurrence relations:*

$$X_N \mathbf{1}_{\{\sigma=N\}} = G_N \mathbf{1}_{\{\sigma=N\}}$$

$$X_n \mathbf{1}_{\{\sigma\geqslant n\}} = G_n \mathbf{1}_{\{\sigma=n\}} + \frac{1}{1+r} \mathbf{1}_{\{\sigma>n\}} \tilde{\boldsymbol{E}}_n X_{n+1}\,.$$

If we write $\omega = (\omega', \omega_{n+1}, \omega'')$ with $\omega' = (\omega_1, \dots, \omega_n)$, then we know in the Binomial model we have

$$\tilde{\boldsymbol{E}}_n X_{n+1}(\omega) = \tilde{\boldsymbol{E}}_n X_{n+1}(\omega') = \tilde{p} X_{n+1}(\omega', 1) + \tilde{q} X_{n+1}(\omega', -1)\,.$$

As before, we will use state processes to find practical algorithms to price securities.

Example 6.43. Let $A, U > 0$. The up-and-rebate option pays the face value A at the first time the stock price exceeds U (up to maturity time N), and nothing otherwise. Find an efficient way to compute the arbitrage free price of this option.

Proposition 6.44. *Let $Y = (Y^1, \dots, Y^d)$ be a d-dimensional process such that for every n we have $Y_{n+1}(\omega) = h_{n+1}(Y_n(\omega), \omega_{n+1})$ for some deterministic function h_{n+1}. Let A_1, ..., $A_N \subseteq \mathbb{R}^d$, with A_N $\mathbb{R}^d$, and define the stopping time σ by*

$$\sigma = \min\{n \in \{0, \dots, N\} \mid Y_n \in A_n\}\,.$$

Let g_0, ...g_N be N deterministic functions on $\mathbb{R}^d$, and consider a security that pays $G_\sigma = g_\sigma(Y_\sigma)$. The arbitrage free price of this security is of the form $V_n \mathbf{1}_{\{\sigma \geqslant n\}} = f_n(Y_n)\mathbf{1}_{\{\sigma \geqslant n\}}$. The functions f_n satisfy the recurrence relation

$$f_N(y) = g_N(y)$$

$$f_n(y) = \mathbf{1}_{\{y \in A_n\}} g_n(y) + \frac{\mathbf{1}_{\{y \notin A_n\}}}{1+r} \Big(\tilde{p} f_{n+1}(h_{n+1}(y, 1)) + \tilde{q} f_{n+1}(h_{n+1}(y, -1)) \Big)$$

6.4. **Optional Sampling.** Consider a market with a few risky assets and a bank.

Question 6.45. *If there is no arbitrage opportunity at time N, can there be arbitrage opportunities at time $n \leqslant N$? How about at finite stopping times?*

Proposition 6.46. *There is no arbitrage opportunity at time N if and only if there is no arbitrage opportunity at any finite stopping time.*

Question 6.47. *Say M is a martingale. We know $\boldsymbol{E} M_n = \boldsymbol{E} M_0$ for all n. Is this also true for stopping times?*

Theorem 6.48 (Doob's optional sampling theorem)**.** *Let τ be a bounded stopping time and M be a martingale. Then $\boldsymbol{E}_n M_\tau = M_{\tau \wedge n}$.*

Proposition 6.49. *Suppose a market admits a risk neutral measure. If X is the wealth of a self-financing portfolio and τ is a finite stopping time such that $X_0 = 0$, and $X_\tau \geqslant 0$, then $X_\tau = 0$.*

Remark 6.50. This is simply an alternate proof of Proposition 6.46.

Question 6.51 (Gamblers ruin)**.** *Suppose* $N = \infty$. *Let* X_n *be i.i.d. random variables with mean* 0, *and let* $S_n = \sum_1^n X_k$. *Let* $\tau = \min\{n \mid S_n = 1\}$. *(It is known that* $\tau < \infty$ *almost surely.) What is* $\boldsymbol{E} S_\tau$*? What is* $\lim_{N \to \infty} \boldsymbol{E} S_{\tau \wedge N}$*?*

6.5. **American Options.** An American option is an option that can be exercised at any time chosen by the holder.

Definition 6.52. Let $G_0, G_1, \dots, G_N$ be an adapted process. An *American option* with *intrinsic value* G is a security that pays G_σ at any finite stopping time σ chosen by the holder.

Example 6.53. An *American put* with strike K is an American option with intrinsic value $(K - S_n)^+$.

Question 6.54. *How do we price an American option? How do we decide when to exercise it? What does it mean to replicate it?*