Lecture 18 (10/13). Please enable your video if you can

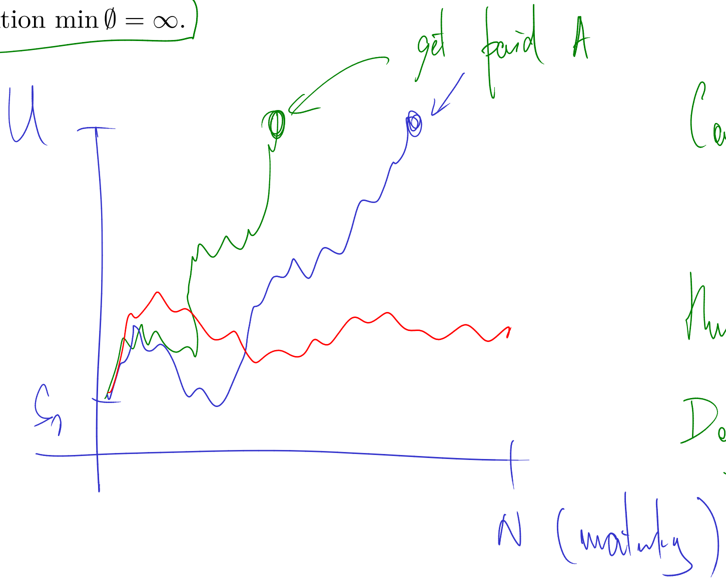
Notation: $a \wedge b = \min\{a, b\}$

$a \vee b = \max\{a, b\}$.

6.3. **Options with random maturity.** Consider the N period binomial model with $0 < d < 1 + r < u$.

Example 6.29 (Up-and-rebate option). Let $A, U > 0$. The up-and-rebate option pays the face value A at the first time the stock price exceeds U (up to maturity time N), and nothing otherwise. Explicitly, let $\tau = \min\{n \leqslant N \mid S_n \geqslant U\}$, and let $\sigma = \tau \wedge N$. The up-and-rebate options pays $A\mathbf{1}_{\tau \leqslant N}$ at the random time σ.

Remark 6.30. By convention $\min \emptyset = \infty$.



Could happen that $\{S_n \leq U$
$\forall n \leq N$.

then $\{n \leq N \mid S_n \geq U\} = \phi$

Define $\min \phi = +\infty$.

Definition 6.31. We say a random variable τ is a *stopping time* if:

(1) $\tau : \Omega \to \{0, \dots, N\} \cup \{\infty\}$
(2) For all $n \leqslant N$, the event $\{\tau \leqslant n\} \in \mathcal{F}_n$.

Remark 6.32. We say τ is a finite stopping time if $\tau < \infty$ almost surely.

Remark 6.33. The second condition above is equivalent to requiring $\{\tau = n\} \in \mathcal{F}_n$ for all n.

$\tau \longrightarrow$ ~~as~~ time we decide to stop playing a game

$\{\tau = n\} \longrightarrow$ event we decided to stop playing at time n.

Require $\{\tau = n\} \in \mathcal{F}_n$ (only uses first n coin tosses).

Question 6.34. *Is* $\tau = 5$ *a stopping time?* Yes: check: ① Range$(\tau) \subseteq \{0, \dots N\} \cup \{\infty\}$ ✓

$\{5\}$

Question 6.35. *Is the first time the stock price hits* U *a stopping time?*

Question 6.36. *Is the last time the stock price hits* U *a stopping time?*

τ = first time stock price exceeds U.

τ = last time " " " U.

$$\tau = \min\{n \le N \mid S_n \ge U\}$$

$$\tau = \max\{n \le N \mid S_n \ge U\}$$

Guess τ is NOT a stopping time Yes

② NTS $\{\tau = n\} \in \mathcal{F}_n \ \forall n$.

$$\{\tau = n\} = \begin{cases} \Omega & n = 5 \\ \phi & \text{OW.} \end{cases}$$

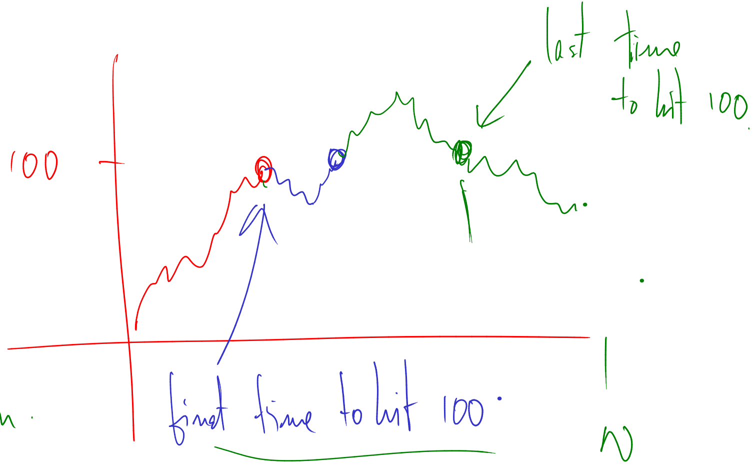
$\Rightarrow \Omega \in \mathcal{F}_n \ \forall n$ & $\phi \in \mathcal{F}_n \ \forall n$

$\Rightarrow \{\tau = n\} \in \mathcal{F}_n \ \forall n$ ✓

Guess τ is a stopping time. Yes.

To see τ is a stopping time $\in \mathcal{F}_n$.

$$\{\tau = n\} = \{S_k < U \ \forall k < n \ \& \ \underbrace{S_n \geq U}_{\in \mathcal{F}_n}\} \in \mathcal{F}_n$$



$$\{\tau = n\} = \{\underbrace{S_k < U \ \forall k > n}_{\text{Requires info on coin tosses after } n \ \notin \mathcal{F}_n!} \ \& \ \underbrace{S_n \geq U}_{\in \mathcal{F}_n}\}$$

need not $\notin \mathcal{F}_n$.

Question 6.37. *If σ and τ are stopping times, is $\sigma \wedge \tau$ a stopping time? How about $\sigma \vee \tau$?*

$\max\{\sigma, \tau\}$

Think about both these

$\min\{\sigma, \tau\}$

- Let G be an adapted process, and σ be a *finite* stopping time.
- Consider a derivative security that pays G_σ at the random time σ.
- Note $G_\sigma = \sum_{n=0}^{N} G_n \mathbf{1}_{\{\sigma=n\}}$
- Let $(X_0, (\Delta_n))$ be a self-financing portfolio, and X_n at time n be the wealth of this portfolio at time n.

(i.e σ is a stopping time & $P\{\sigma < \infty\} = 1$.)

($G_\sigma = G_n$ when $\sigma = n$).

Definition 6.38. A self-financing portfolio with wealth process X is a replicating strategy if $\boxed{X_\sigma = G_\sigma}$.

Theorem 6.39. *The security with payoff G_σ (at the stopping time σ) can be replicated. The arbitrage free price is given by*

$$X_n \mathbf{1}_{\{\sigma \geqslant n\}} = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n (D_\sigma G_\sigma \mathbf{1}_{\{\sigma \geqslant n\}})$$

Remark 6.40. The only thing required for the proof of Theorem 6.39 is the fact that X_n is the wealth of a self-financing portfolio if and only if $D_n X_n$ is a $\tilde{\boldsymbol{P}}$ martingale.

Proof: Let $Z = D_\sigma G_\sigma$ (some R.V.)

Let $X_n = \frac{1}{D_n} \tilde{E}_n (Z) = \frac{1}{D_n} \tilde{E}_n (D_\sigma G_\sigma)$

- Claim ① X_n is the wealth of a self financing portfolio.
- Claim ② $X_\tau = G_\tau$.

(Note: Claim ① + ② $\Rightarrow$ X is a replicating portfolio of the security with payoff G_τ at time τ.

$\Rightarrow$ AFP of security at time $n \le \tau$ is X_n.

Note
$$\mathbb{1}_{\{n=\tau\}} X_n = \mathbb{1}_{\{n=\tau\}} \frac{1}{D_n} \tilde{E}_n\left(D_\tau G_\tau\right)$$

(Note $\{\tau = n\} \in \mathcal{F}_n$

since τ is a stopping time

$\Rightarrow \mathbb{1}_{\{\tau = n\}}$ is $\mathcal{F}_n$ meas)

$$= \frac{1}{D_n} \tilde{E}_n \left(\mathbb{1}_{\{n=\tau\}} D_\tau G_\tau \right)$$

$$= \frac{1}{D_n} \tilde{E}_n \left(\mathbb{1}_{\{n=\tau\}} D_n G_n \right)$$

$\therefore \quad \mathbb{1}_{\{n=\tau\}} X_n = \frac{1}{D_n} \tilde{E}_n \left(\mathbb{1}_{\{n=\tau\}} D_n G_n \right)$

$$= \mathbb{1}_{\{n=\tau\}} G_n$$

$\Rightarrow X_\tau = G_\tau$

$(\Rightarrow$ Claim 2$)$.

Pf of claim 1: NTS X_n = wealth of self fin port

Enough to show $D_n X_n$ is a $\tilde{P}$ mg

But $D_n X_n = \tilde{E}_n Z = \tilde{E}_n (\overbrace{D_T G_T})$

$$\tilde{E}_n(D_{n+1}X_{n+1}) = \tilde{E}_n\left(\tilde{E}_{n+1}(D_T G_T)\right) \overset{\text{tower}}{=} \tilde{E}_n(D_T G_T) = \underline{D_n X_n}$$

QED.