Lecture 17 (10/11). Please enable video if you can

(1) $V_n = \frac{1}{D_n} \tilde{E}_n (D_N V_N)$ $\leftarrow$ takes $O(2^N)$ time to compute.

(2) If $\boxed{V_N = g(S_N)}$, $f_n(x) = \frac{1}{1+r}\left(\tilde{p} f_{n+1}(ux) + \tilde{q} f_{n+1}(dx)\right)$

$\boxed{\& \ f_N(x) = g(x).}$

$V_n = f_n(S_n)$ $\longrightarrow$ takes $\underline{O(N^3)}$ time to compute

(3) If V_N is not in this form (Eg. Knockout options), find fast algorithms

Definition 6.20. We say a d-dimensional process $Y = (Y^1, \dots, Y^d)$ process is a *state process* if for any security with maturity $m \leqslant N$, and payoff of the form $V_m = f_m(Y_m)$ for some (non-random) function f_m, the arbitrage free price must also be of the form $V_n = f_n(Y_n)$ for some (non-random) function f_n.

Remark 6.21. For state processes given f_N, we typically find f_n by backward induction. The number of computations at time n is of order Range(Y_n).

Remark 6.22. The fact that S_n is $\boxed{\text{Markov}}$ (under $\tilde{\boldsymbol{P}}$) implies that it is a state process.

Convention: superscript $\longrightarrow$ coordinate

Sub script $\longrightarrow$ time.

State process:

If payoff is a fn of the state then AFP is also a fn of the state

Theorem 6.23. *Let $Y = (Y^1, \ldots, Y^d)$ be a d-dimensional process. Suppose we can find functions $g_1, \ldots, g_N$ such that $Y_{n+1}(\omega) = g_{n+1}(Y_n(\omega), \omega_{n+1})$. Then Y is a state process.*

① Pf of thm: Consider a sec. that pays $\overbrace{f_N(Y_N)}^{V_N.}$ at time N.

NTS $\forall\ n \leq N$, AFP at time n is some fn of Y_n.

② Say AFP at time $n+1$ is $f_{n+1}(Y_{n+1})$. [True for $n = N-1$]

NTS AFP at time n is $f_n(Y_n)$ for some fn f_n that I will find.

Know AFP at time n $\Rightarrow$ $V_n = \frac{1}{1+r}\tilde{E}_n \underline{V_{n+1}}$

$$\Rightarrow V_n = \frac{1}{1+r}\tilde{E}_n f_{n+1}(\underbrace{Y_{n+1}}) = \frac{1}{1+r}\tilde{E}_n f_{n+1}\Big(g_{n+1}\big(\underbrace{Y_n}_{\mathcal{F}_n\text{-meas}}, \underbrace{\omega_{n+1}}_{\text{ind of } \mathcal{F}_n}\big)\Big)$$

$$\overset{\text{indep lemma}}{=} \frac{1}{1+r}\Big(\tilde{p}\, f_{n+1}\big(g_{n+1}(Y_n, +1)\big) + \tilde{q}\, f_{n+1}\big(g_{n+1}(Y_n, -1)\big)\Big)$$

$$\Rightarrow V_n = f_n(Y_n) \text{ where } \overbrace{\boxed{f_n(y) = \frac{1}{1+r}\left(\tilde{p}\, f_{n+1}(g_{n+1}(y,+1)) + \tilde{q}\, f_{n+1}(g_{n+1}(y,-1))\right)}}^{\text{Some fn of } Y_n}$$

QED

Note: Gives a recurrence rel to find f_n in terms of f_{n+1}

computations $\approx O(\text{Range}(Y_n))$

Question 6.24. *Is $Y_n = S_n$ a state process?* ← Yes (started with this)

Question 6.25. *Is $Y_n = \max_{k\leqslant n} S_k$ a state process?* ← Not a state process.

Question 6.26. *Is $Y_n = (S_n, \max_{k\leqslant n} S_n)$ a state process?*

$$M_n = \max_{k\leq n} S_k. \longrightarrow \text{state process?} \leftarrow \text{NO}$$

(2) $Y_n = (S_n, M_n) \longrightarrow$ state process? ← Yes.

↳ To check this is a state process only need to write

Y_{n+1} as some fn of Y_n & ω_{n+1}, Note $Y'_{n+1} = S_{n+1} = S_n \left(u \, \mathbb{1}_{\omega_{n+1}=1} + d \, \mathbb{1}_{\omega_{n+1}=-1} \right)$

$$= \underbrace{Y'_n \left(u \, \mathbb{1}_{\omega_{n+1}=1} + d \, \mathbb{1}_{\omega_{n+1}=-1} \right)}_{\text{some fn of } Y_n \text{ \& } \omega_{n+1}.}$$

(Recall $\mathbb{1}_A \to$ ~~w~~ RV

$$\mathbb{1}_A = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$
)

$$Y^2_{n+1} = M_{n+1} = \max_{k \le n+1} S_k = \max\left(S_{n+1} , \max_{k \le n} S_k \right)$$

$$= \max \underbrace{\left(Y^1_n \left(\mathbb{1}_{\{\omega_{n+1} = 1\}} u + \mathbb{1}_{\omega_{n+1} = -1} d \right) , Y^2_n \right)}_{\text{Same fn of } Y_n \text{ \& } \omega_{n+1}}$$

Question 6.27. *Let* $A_n = \sum_0^n S_k$. *Is* A_n *a state process?* ← NO

Question 6.28. *Is* $Y_n = (S_n, A_n)$ *a state process?*

Yes (related to HW)

$S_0 = 1$

$\text{Range}(S_1) = \{u, d\}$

$\text{Rge}(S_2) = \{u^2, ud, d^2\}$

$\text{Rge}(A_1) = \{1+u, 1+d\}$

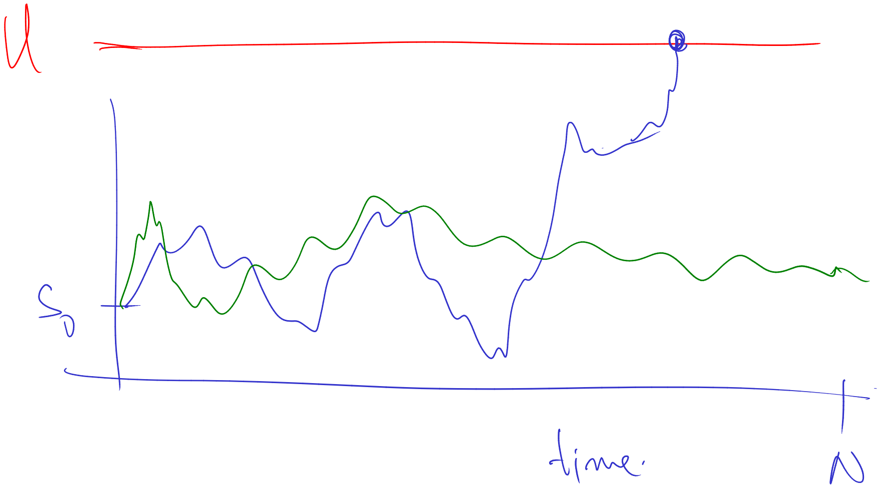
$\text{Rge}(A_2) = \{1+u+u^2, 1+u+ud, 1+d+ud, 1+d+d^2\}$

$\text{Rge}(A_3) =$ 8 vals

6.3. **Options with random maturity.** Consider the N period binomial model with $0 < d < 1 + r < u$.

Example 6.29 (Up-and-rebate option). Let $A, U > 0$. The up-and-rebate option pays the face value A at the first time the stock price exceeds U (up to maturity time N), and nothing otherwise. Explicitly, let $\tau = \min\{n \leqslant N \mid S_n \geqslant U\}$, and let $\sigma = \tau \wedge N$. The up-and-rebate options pays $A\mathbf{1}_{\tau \leqslant N}$ at the random time σ.

Remark 6.30. By convention $\min \emptyset = \infty$.



Definition 6.31. We say a random variable τ is a *stopping time* if:

(1) $\tau\colon \Omega \to \{0, \dots, N\} \cup \infty$
(2) For all $n \leqslant N$, the event $\{\tau \leqslant n\} \in \mathcal{F}_n$.

Remark 6.32. We say τ is a finite stopping time if $\tau < \infty$ almost surely.

Remark 6.33. The second condition above is equivalent to requiring $\{\tau = n\} \in \mathcal{F}_n$ for all n.