HW 4 Q4e]

$P \longrightarrow p$ iid coins $\begin{cases} \text{Heads with prob } p_1 \\ \text{Tails '' '' } r_1 \end{cases}$

$\tilde{P} \longrightarrow \tilde{p}$ iid coins $\begin{cases} \text{Heads with prob } \tilde{p}_1 = \frac{(1+r)-d}{u-d} \\ \text{Tails with prob } \tilde{q}_1 = \frac{u-(1+r)}{u-d} \end{cases}$

Q: Want $\boxed{\tilde{p} = Z_N\, p}$ (find a formula for Z)

Choose $Z_N = \dfrac{\tilde{P}}{P}$. & find formulae for P & $\tilde{P}$.

1) Formula for P: $P(\omega) = \underbrace{P(\omega_1, \omega_2, \ldots \omega_N)}$

$= \left(p_1^{\#\text{ heads}}\right) \cdot \left(q_1^{\#\text{ tails}}\right) \leftarrow$

all heads : p_1^N

$\underbrace{1^{st}}$ tails, rest heads: $q_1 p_1^{N-1}$

$\left(\begin{array}{l} \#\text{ heads} = \#i \ni \omega_i = +1 \\ \& \#\text{ tails} = \#i \ni \omega_i = -1 \end{array}\right)$

(2) $\tilde{p}(\omega) = (\tilde{p}_1)^{\# \text{heads}} \; \tilde{q}_1^{\# \text{tails}}$

(3) $\Rightarrow Z_N(\omega) = \dfrac{\tilde{p}(\omega)}{p(\omega)} = \underbrace{\left(\dfrac{\tilde{p}_1}{p_1}\right)^{\# \text{heads}} \cdot \left(\dfrac{\tilde{q}_1}{q_1}\right)^{\# \text{tails}}}$

(4) Formula for $Z_n(\omega) = \underbrace{E_n Z_N}$ (Knows this is $\mathcal{F}_n$ meas)

(5) $\boxed{\text{Guess}}$ $Z_n(\omega) = \left(\dfrac{\tilde{p}_1}{p_1}\right)^{\# \text{heads up to time } n} \cdot \left(\dfrac{\tilde{q}_1}{q_1}\right)^{\# \text{tails up to time } n.}$

⑤ Find formula for Z_{N-1}

$\omega = (\underbrace{\omega_1 \cdots \omega_{N-1}}, \underset{=}{\omega_N})$

Compute $E_{N-1} Z_N$

$$\underbrace{E_{N-1} Z_N(\omega)} = E_{N-1}\left[\left(\frac{\tilde{p}_1}{p_1}\right)^{H_{N-1}} \left(\frac{\tilde{q}_1}{q_1}\right)^{T_{N-1}} X_N(\omega) \right]$$

$\mathcal{F}_{N-1}$ — meas.

H_n = # heads up to time n.
T_n = " tails " " " n.

$$X_N(\omega) = \begin{cases} \left(\frac{\tilde{p}_1}{p_1}\right) & \omega_N = 1 \\ \left(\frac{\tilde{q}_1}{q_1}\right) & \omega_N = -1 \end{cases}$$

$$= \underbrace{\left(\frac{\tilde{p}_1}{p_1}\right)^{H_{N-1}}} \underbrace{\left(\frac{\tilde{q}_1}{q_1}\right)^{T_{N-1}}} E_{N-1} \tilde{X}_N$$

ind of $\mathcal{F}_{N-1}$

$$= \left(\frac{\tilde{p}_1}{p_1}\right)^{H_{N-1}} \left(\frac{\tilde{q}_1}{q_1}\right)^{T_{N-1}} E X_N$$

$$= \left(\frac{\tilde{p}_1}{p_1}\right)^{H_{N-1}} \left(\frac{\tilde{q}_1}{q_1}\right)^{T_{N-1}} \underbrace{\left(\left(\frac{\tilde{p}_1}{p_1}\right) \cdot p_1 + \left(\frac{\tilde{q}_1}{q_1}\right) q_1 \right)}_{=1}$$

Guess is correct. $\mathcal{A}$ is a proof for $n = N-1$

that going.

Q5 (b) $\tilde{p}(\omega) =$ guess $(n+1)^{th}$ coin lands heads with prob

$$\boxed{\frac{1 + r_n(\omega) - d_n}{u_n - d_n}}$$

depends on prev cointosses.

$$S_1 = \begin{cases} u_0 S_0 & \leftarrow \text{prob } p_1 \\ d_0 S_0 & \leftarrow \text{prob } q_1 \end{cases} \quad \text{under } P.$$

RNP of 1st coin $\begin{cases} \dfrac{1+r_0-d_0}{u_0-d_0} & \text{heads} \\ \dfrac{u_0-(1+r_0)}{u_0-d_0} & \text{tails.} \end{cases}$

Write $\tilde{p}(\omega) = \underbrace{\tilde{p}_1(\omega_1)} \; \underbrace{\tilde{p}_2(\omega_1, \omega_2)} \; \tilde{p}_3(\omega_1, \omega_2, \omega_3) \cdots \tilde{p}_N(\omega_1, \cdots \omega_N)$

RNP of 1^{st} coin

RNP 2^{nd} coin

$$p_1(\omega_1) = \begin{cases} \dfrac{1+r_0-d_0}{u_0-d_0} & \omega_1 = 1 \\[2ex] \dfrac{u_0-(1+r_0)}{u_0-d_0} & \omega_1 = -1 \end{cases}$$

$$p_2(\omega_1,\omega_2) = \begin{cases} \dfrac{1+r_1(\omega_1)-d_1(\omega_1)}{u_1(\omega_1)-d_1(\omega_1)} & \omega_2 = +1 \\[2ex] \big(u_1(\omega_1)-(1+r_1)(\omega_1)\big)\big/\big(u_1(\omega_1)-d_1(\omega_1)\big) & \omega_2 = -1 \end{cases}$$

Say $N=2$. $\tilde{p}(\omega) = \tilde{p}(+1, -1)$

Say $\omega = (+1, -1)$

$$= \underbrace{\text{RNP of 1st coin heads}}_{\left(\frac{1+r_0-d_0}{u_0-d_0}\right)} \;\&\; \underbrace{\text{2nd coin tails}}_{\left(\frac{u_1(1)-(1+r_1(1))}{u_1(1)-d_1(1)}\right)}$$