Lec 14 (10/4). Please enable your video if you can

Goal: Prove Thm 6.7.

Binomial model d, r, u

$$S_{n+1} = \begin{cases} u S_n & n+1^{\text{th}} \text{ coin heads} \\ d S_n & \text{" " tails.} \end{cases}$$

$\boxed{0 < d < 1+r < u.}$

Know $\exists$ a equiv measure $\tilde{P}$ + $D_n S_n$ is a mg under $\tilde{P}$.

$\left(D_n = \frac{1}{(1+r)^n} \text{ discount factor} \right)$

$\boxed{\text{Thm 6.7}}$ Let X = wealth process of any investor.

X is the wealth of a $\boxed{\text{self financing}}$ portfolio

$\iff D_n X_n$ is a $\tilde{P}$ mg "

Recall: Self fin: (No external cash flow & no looking in future)

$\hookrightarrow$ X_n is adapted

& $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$

& need Δ_n to be adapted

trading strat.

n

Intuition :

Casino

Fair game = Mg.

Stock market, 1 stock

M.M interest rate r.

→ Get 1 secure asset (M.M.)

Pays interest.

→ Discount cash & pretend $r = 0$

$$\text{Exp yield from disc Stock} = \left[\frac{u p + q d}{1+r} \right]$$

→ But : under $\tilde{P}$: Exp yield from disc Stock : $\dfrac{u\tilde{p} + d\tilde{q}}{1+r} = 1$

Proof of Theorem 6.7 part 1. Suppose X_n is the wealth of a self-financing portfolio. Need to show $D_n X_n$ is a martingale under $\tilde{\boldsymbol{P}}$.

assume

NTS.

Pf: Know $\exists$ an adapted trading strat $\ni$ $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$

(def of self fin).

Compute $\tilde{E}_n (D_{n+1} X_{n+1})$ $\left(\underline{\underline{\text{Want}}} \quad D_n X_n \right)$

$$D_n = \frac{1}{(1+r)^n}.$$

$$D_{n+1}(1+r) = D_n.$$

$$\tilde{E}_n (D_{n+1} X_{n+1}) = \tilde{E}_n \Big(\Delta_n D_{n+1} S_{n+1} + \underbrace{D_{n+1}(1+r)(X_n - \Delta_n S_n)}_{\mathcal{F}_n\text{-meas}} \Big)$$

$\mathcal{F}_n$ - meas

$$= \Delta_n \tilde{E}_n\left(D_{n+1} S_{n+1}\right) + D_n\left(X_n - \Delta_n S_n\right)$$

$$= \Delta_n \cancel{D_n} S_n + D_n X_n - \cancel{D_n \Delta_n} S_n$$

($\because$ $D_n S_n$ is a $\tilde{P}$ mg).

$$= D_n X_n$$

QED.

Proof of Theorem 6.7 part 2. Suppose $D_n X_n$ is a martingale under $\tilde{\boldsymbol{P}}$. Need to show X_n is the wealth of a self-financing portfolio.

Assume $D_n X_n$ is $\tilde{P}$ mg.

NTF Δ_n such that $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$

& Δ_n is $\mathcal{F}_n$-meas.

Look at $\omega = (\underbrace{\omega'}_{\text{first } n}, \omega_{n+1}, \underbrace{\omega''}_{\text{last few}})$

$\omega' = (\omega_1, \dots \omega_n)$

$\omega'' = (\omega_{n+2}, \dots \omega_N)$.

$X_{n+1}(\omega) = X_{n+1}(\omega', \omega_{n+1}, \cancel{x})$

Look at $\begin{pmatrix} X_{n+1}(\omega', 1) \\ X_{n+1}(\omega', -1) \end{pmatrix} \in \mathbb{R}^2.$

<u>Note</u> $\Big\{ \underbrace{\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \Big\}$ are linearly Ind in $\mathbb{R}^2$.

Write X_{n+1} as a lin comb of

Know $\exists\ \Delta_n(\omega')$ & $\Gamma_n(\omega')$ such that

$$\begin{pmatrix} X_{n+1}(\omega', 1) \\ X_{n+1}(\omega', -1) \end{pmatrix} = \Delta_n(\omega') \, S_n(\omega') \begin{pmatrix} u \\ d \end{pmatrix} + \Gamma_n(\omega') \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$S_n(\omega') \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} S_{n+1}(\omega', 1) \\ S_{n+1}(\omega', -1) \end{pmatrix}$$

$$\Rightarrow X_{n+1}(\omega) = \Delta_n(\omega') \, S_{n+1}(\omega) + \Gamma_n(\omega')$$

$\Rightarrow X_{n+1} = \Delta_n S_{n+1} + \Gamma_n$ for some adapted Δ_n & Γ_n

NTS: $\Gamma_n = \underline{(X_n - \Delta_n S_n)(1+r)}$ *

Pf: Know $D_n X_n$ is a $\tilde{P}$ mg.

Know $D_{n+1} X_{n+1} = D_{n+1} \Delta_n S_{n+1} + D_{n+1} \Gamma_n$.

$D_{n+1} = \frac{1}{(1+r)^{n+1}}$

$$\Rightarrow D_n X_n = \tilde{E}_n (D_{n+1} X_{n+1}) = \Delta_n \underbrace{\tilde{E}_n (D_{n+1} S_{n+1})}_{D_n S_n} + D_{n+1} \Gamma_n$$

$$\Rightarrow \quad D_n X_n = \underline{\Delta_n D_n S_n} + D_{n+1} r_n .$$

$$\Rightarrow \quad r_n = (1+r)\left(X_n - \Delta_n S_n\right)$$ which is what we wanted

QED.