Lecture 13 (10/1). Please enable video if you can.

- Consider an investor that starts with X_0 wealth, which he divides between cash and the stock.
- If he has Δ_0 shares of stock at time 0, then $X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$.
- We allow the investor to trade at time 1 and hold Δ_1 shares.
- Δ_1 may be random, *but must be $\mathcal{F}_1$-measurable.*
- Continuing further, we see $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$.
- *Both X and Δ are adapted processes.*

RNM is an equivalent measure under which $D_n S_n$ is a $\tilde{P}$ mg

Definition 6.6. A *self-financing portfolio* is a portfolio whose wealth evolves according to

$$\boxed{X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)\,,}$$

for some $\boxed{\text{adapted}}$ process Δ_n.

Theorem 6.7. *Let $d < 1 + r < u$, and $\tilde{\boldsymbol{P}}$ be the risk neutral measure, and X_n represent the wealth of a portfolio at time n. The portfolio is self-financing portfolio if and only if the discounted wealth $D_n X_n$ is a martingale under $\tilde{\boldsymbol{P}}$.*

Remark 6.8. The only thing we will use in this proof is that $D_n S_n$ is a martingale under $\tilde{\boldsymbol{P}}$. The interest rate r can be a *random* adapted process. It is also not special to the binomial model – it works for any model for which there is a risk neutral measure.

IOU a proof of this theorem.

Before proving Theorem 6.7, we consider a few consequences:

Theorem 6.9. *The multi-period binomial model is arbitrage free if and only if* $d < 1 + r < u$.

Definition 6.10. We say the market is arbitrage free if for any self financing portfolio with wealth process X, we have: $X_0 = 0$ and $X_N \geqslant 0$ implies $X_N = 0$ almost surely.

Remark 6.11. The first fundamental theorem of asset pricing states that a risk neutral measure exists if and only if the market is arbitrage free. (We will prove this in more generality later.)

Pf: (1) $1+r \leq d$: Have arb (borrow cash & buy stock → arbitrage)

(2) $1+r \geq u$: Have arb (short stock & bank cash)

(3) $d < 1+r < u$. NTS $\nexists$ arb.

Know a RNM Exists. Call it $\tilde{P}$.

Say $X_0 = 0$. X = wealth process of a self fin portfolio.

$\boxed{\text{Say } X_N \geq 0}$ NTS: $X_N = 0$ almost surely.

Pf: Know (Thm 6.7) $D_n X_n$ is a $\tilde{P}$ mg.

$\Rightarrow D_0 X_0 = \tilde{E}(D_N X_N)$ ($\because D_n X_n$)

$\Rightarrow \tilde{E}(D_N X_N) = 0$

$\Rightarrow$ Know ① $\tilde{E}(D_N X_N) = 0$ ② $D_N X_N \geq 0$ $\to$ Can only happen if $D_N X_N = 0$ ($\tilde{P}$ a.s.)

Since $D_N = (1+r)^{-N} \Rightarrow X_N = 0 \quad (\tilde{P} \text{ a.s.})$

$\Rightarrow X_N = 0 \quad (P \text{ a.s.}).$

$\Rightarrow$ No arb QED.

Theorem 6.12 (Risk Neutral Pricing Formula). *Let $\boxed{d < 1 + r < u,}$ and V_N be an $\mathcal{F}_N$ measurable random variable. Consider a security that pays V_N at maturity time N. For any $n \leqslant N$, the arbitrage free price of this security is given by*

$$V_n = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n(D_N V_N) . \quad = (1+r)^{n-N} \tilde{E} V_N .$$

($V_N \to$ payoff of a security, e.g $V_N = (S_N - K)^+$)

Pf: Replication $\to$ Will find a self financing portfolio with wealth process X such that $X_N = V_N$.

$\Rightarrow$ AFP of security at time n $= X_n$.

Find X_n: let $X_N = V_N$.

For $n \le N$, let $\underline{X_n} = \frac{1}{D_n} \tilde{E}_n(D_N V_N) = \frac{1}{D_n} \tilde{E}_n(D_N X_N)$

NTS: X_n = wealth of a self fin pf

Thm 6.7, Enough to show $D_n X_n$ is a $\tilde{P}$ mg.

Pf: $\tilde{E}_n(D_{n+1} X_{n+1}) = \tilde{E}_n\left(D_{n+1} \cdot \frac{1}{D_{n+1}} \tilde{E}_{n+1}(D_N X_N)\right)$

$$= \tilde{E}_n\left(\tilde{E}_{n+1}(D_N X_N)\right) \underset{\text{tower}}{=} \tilde{E}_n(D_N X_N)$$

$= D_n X_n$ (def of X_n)

$\Rightarrow D_n X_n$ is a $\tilde{P}$ mg

$\Rightarrow X_n$ = wealth of a self fin Pf (Thm 6.7)

Since $X_N = V_N \Rightarrow X_n$ = AFP of sec at time n.

$\Rightarrow$ AFP of sec $= X_n = \frac{1}{D_n} \tilde{E}_n (D_N X_N)$. QED.

Remark 6.13. The replicating strategy can be found by backward induction. Let $\omega = (\omega', \omega_{n+1}, \omega'')$. Then

$$\omega' = (\omega_1, \cdots \omega_n)$$

$$\omega'' = (\omega_{n+2}, \cdots \omega_N)$$

$$\Delta_n(\omega) = \frac{V_{n+1}(\omega', 1, \omega'') - V_{n+1}(\omega', -1, \omega'')}{(u-d)S_n(\omega)} = \frac{V_{n+1}(\omega', 1) - V_{n+1}(\omega', -1)}{(u-d)S_n(\omega)}$$

Pf:

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$$

(Wealth of our rep pf)

$$X_n = V_n \quad (\text{Rep pf}).$$

1. $\omega_{n+1} = +1$: $V_{n+1}(\omega', +1, *) = \Delta_n(\omega')\, u\, S_n(\omega') + (1+r)\left(V_n(\omega') - \Delta_n(\omega') S_n(\omega')\right)$

2. $\omega_{n+1} = -1$: $V_{n+1}(\omega', -1, *) = \Delta_n(\omega')\, d\, S_n(\omega') + (1+r)\left[V_n(\omega') - \Delta_n(\omega') S_n(\omega')\right]$

u

$$① - ② \Rightarrow V_{n+1}(\omega', +1) - V_{n+1}(\omega', -1) = \Delta_n(\omega') \, S_n(\omega') \, (u-d)$$

$$\Rightarrow \Delta_n(\omega') = \frac{V_{n+1}(\omega', +) - V_{n+1}(\omega', -1)}{S_n(\omega') \, (u-d)}.$$

QED