Q HW4 $\boxed{Q4}$.

Z_N a $\mathcal{F}_N$ meas RV, $E Z_N = 1$, $Z_N > 0$

$\tilde{P}(A)$ define $= \sum_{\omega \in A} Z_N(\omega)\, p(\omega)$

(or set $\tilde{p}(\omega) = Z_N(\omega)\, p(\omega)$

① $\tilde{E} X = E(Z_N X)$

② Let $Z_n = E_n Z_N$. Then Z is a mg

$$\left(\text{Pf: } E_n Z_{n+1} = E_n E_{n+1} Z_N \overset{\text{tower}}{=} E_n Z_N \right)$$

③ Claim: $Z_n > 0$

Pf: $Z_n(\omega) = E_n Z_N(\omega)$

$$= \frac{1}{P(\Pi_n(\omega))} \sum_{\omega' \in \Pi_n(\omega)} \overbrace{Z_N(\omega)}^{>0} \, p(\omega) > 0.$$

④ Claim $\underset{\sim}{\tilde{E}_n} X = \dfrac{1}{Z_n} E_n(Z_N X)$.

Pf: ⓐ Show $\dfrac{1}{\underset{\sim}{Z_n}} E_n(Z_N X)$ is $\mathcal{F}_n$ meas (True).

ⓑ Show $\forall A \in \mathcal{F}_n$,

$$\sum_{\omega \in A} \left[\frac{1}{\underset{\sim}{Z_n(\omega)}} E_n Z_N X(\omega) \underset{\sim}{\tilde{p}(\omega)} \right] = \sum_{\omega \in A} X(\omega) \tilde{p}(\omega)$$

Want $\forall A \in \mathcal{F}_n$, $\sum_{\omega \in A} \tilde{E}_n X(\omega)\, \tilde{p}(\omega) = \sum_{\omega \in A} X(\omega)\, \tilde{p}(\omega)$

(Def of $\tilde{E}_n$)

i.e. Want $\forall A \in \mathcal{F}_n$, $\sum_{\omega \in A} \tilde{E}_n X(\omega)\, Z_N(\omega)\, p(\omega) = \sum_{\omega \in A} X(\omega)\, Z_N(\omega)\, p(\omega)$

① $RHS = \sum_{\omega \in A} E_n(X Z_N)(\omega)\, p(\omega)$ $\quad (\because A \in \mathcal{F}_n)$

② $$LHS = \sum_{\omega \in A} \tilde{E}_n X(\omega)\, Z_N(\omega)\, p(\omega)$$

$$= \sum_{\omega \in A} E_n\left[(\tilde{E}_n X)\, Z_N \right](\omega)\, p(\omega) \qquad (\because A \in \mathcal{F}_n)$$

$$= \sum_{\omega \in A} \tilde{E}_n X(\omega)\, E_n Z_N(\omega)\, p(\omega)$$

$$= \sum_{\omega \in A} \tilde{E}_n X(\omega)\, Z_n(\omega)\, p(\omega).$$

Since this holds $\forall A \in \mathcal{F}_m$, must have

$$\boxed{E_m(X Z_N) = Z_m \tilde{E}_m X}$$

QED.

4e Eg: $P \longrightarrow$ std meas Binom model.

iid coin tosses $\begin{cases} \to p_1 \\ \to q_1 \end{cases}$

$\tilde{P} \longrightarrow$ R.N. M iid coin tosses $\begin{cases} \to \tilde{p}_1 \\ \to \tilde{q}_1 \end{cases}$

(a) $$p(\underline{\omega}) = p(\omega_1, \dots \omega_N) = p_1^{\#\text{heads}} \, q_1^{\#\text{tails}}$$

$$\tilde{p}(\underline{\omega}) = \tilde{p}(\qquad) = \tilde{p}_1^{\#\text{heads}} \, \tilde{q}_1^{\#\text{tails}}$$

$$\Rightarrow Z_N(\underline{\omega}) = \frac{\tilde{p}(\omega)}{p(\omega)} = \left(\frac{\tilde{p}_1}{p_1}\right)^{\#\text{heads}} \left(\frac{\tilde{q}_1}{q_1}\right)^{\#\text{tails.}}$$

2020 Midterm Q1.

Given

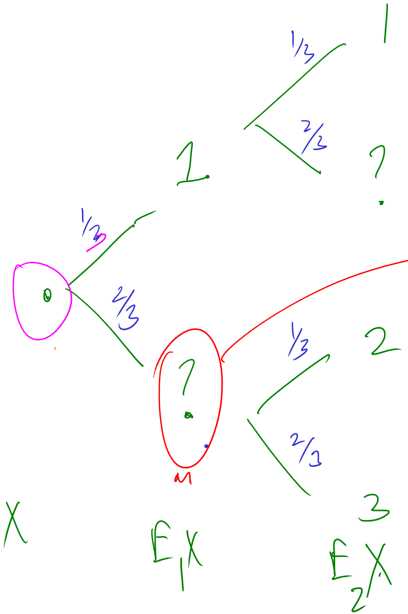
$$E_1 X(1, \dots, 1) = 1$$

$$E_2 X(1, \dots, 1) = 1$$

$$E_2 X(-1, 1, \dots, 1) = 2$$

$$E_2 X(-1, -1, 1, \dots) = 3.$$

Switch :



Has to be $\frac{1}{3} \cdot 2 + \frac{2}{3} \cdot 3$

$\frac{1}{3} \cdot 1 + \frac{2}{3} \cdot (?)$

Writing soln

① Not E_n only depends on 1^{st} n coin tosses

So write $E_1 X(1)$ for $E_1 X(1, \ldots)$.

$E_2 X(1,-1)$ for $E_2 X(1,-1, \ldots)$

② $E_1 X(-1) = \frac{1}{3} E_2 X(-1, 1) + \frac{2}{3} E_2 X(-1,-1)$

$= 2\frac{2}{3} = \frac{8}{3}$

③ $EX = \frac{1}{3} E_1 X(1) + \frac{2}{3} E_1 X(-1) =$

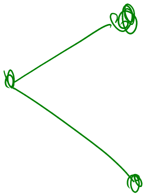
$\boxed{\text{HW 4 Q5}}$

u, d & r $\longrightarrow$ random

u & d adapted (u_n is $\mathcal{F}_n$ - meas

r predictable d_n is $\mathcal{F}_n$ - meas

r_{n+1} is $\mathcal{F}_n$ - meas.)



$n=1$ $\rightarrow$ Knew interest rate at time 0

$$S_{n+1}(\omega) = \begin{cases} u_n(\omega)\, S_n(\omega) & \omega_{n+1} = 1 \\ d_n(\omega)\, S_n(\omega) & \omega_{n+1} = -1 \end{cases}$$

$$D_{n+1} = \frac{D_n(\omega)}{1 + r_n(\omega)}.$$ Disc factor.

Q: Find $\tilde{P}$ so that $D_n S_n$ is a $\tilde{P}$ mg.

Guess: $\tilde{P}(\omega_1 = 1)$ should depend only on r_0, u_1, d_1

not r_1, u_2, etc.

$\tilde{P}(\omega_1 = -1)$ " " " "

$\tilde{P}(\omega_2 = 1)$ & $\tilde{P}(\omega_2 = -1)$ should depend on $\underbrace{\overbrace{r_1, u_2, d_2}}$.

could depend on ω_1, ω_2
but not $\underline{\omega_3 \text{ onwards}}$

Guess: $\tilde{p}(\omega) = \tilde{p}_1(\omega_1)\, \tilde{p}_2(\omega_1, \omega_2)\, \tilde{p}_3(\omega_1, \omega_2, \omega_3) \cdots$
$\cdots \tilde{p}_N(\omega_1, \ldots \omega_N)$.

$\hookrightarrow$ i.e. $\tilde{p}_{n+1}(\omega_1, \ldots \omega_n, +1)$ = RNP of $(n+1)^{th}$ coin being heads given the first n coin tosses are $\omega_1, \ldots \omega_n$.

Finding: $\tilde{p}_n$'s.

Let $X_{n+1}(\omega) = \begin{cases} u & \text{if } \omega_{n+1} = 1 \\ d & \text{if } \omega_{n+1} = -1 \end{cases}$

Want $D_n S_n$ to be a $\tilde{P}$ mg.

Want $\tilde{E}_n\big(\underbrace{D_{n+1}}_{\mathcal{F}_n\text{-meas.}} S_{n+1}\big) \overset{\text{Want}}{=} D_n S_n$

$\Leftrightarrow$ Want

$$D_{n+1} S_n \tilde{E}_n\left(X_{n+1}\right) \overset{\text{Want}}{=} D_n S_n$$

$\Leftrightarrow$ Want $\underbrace{\tilde{E}_n X_{n+1}} = \dfrac{D_n}{D_{n+1}} = \underbrace{(1 + r_n)}$ $r_n(\omega)$

[Guess $\widetilde{E}_n X_{n+1}(\omega) = u_{n+1}(\omega)\widetilde{P}\left(X_{n+1} = 1, \text{ given first } n \text{ coin tosses are } \omega_1, \omega_2 \ldots \omega_n\right)$

$+ d_{n+1}(\omega)\widetilde{P}\left(X_{n+1} = -1, \text{ given } \ldots\right)$

If guess is true: $\widetilde{E}_n X_{n+1}(\omega) = u_{n+1}\widetilde{P}_{n+1}(\omega_1, \ldots \omega_n, +1, \ast \ldots)$

$+ d_{n+1}\widetilde{P}_{n+1}(\omega_1, \ldots \omega_n, -1, \ldots)$

W at this $= 1 + r_n(\omega)$

i.e. $1 + r_n = u_{n+1} \tilde{p}_{n+1}(\omega_1 \ldots \omega_n, 1) - d_{n+1} \tilde{p}_{n+1}(\omega_1 \ldots \omega_n, -1)$

$$\boxed{\Rightarrow \tilde{p}_{n+1}(\omega_1 \ldots \omega_n, 1) = \frac{(1 + r_n) - d_{n+1}}{u_{n+1} - d_{n+1}}}$$