

Lecture 12 (9/27). Please enable your video if you can.

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6. The multi-period binomial model

6.1. Risk Neutral Pricing.

- In the multi-period binomial model we assume $\Omega = \{\pm 1\}^N$ corresponds to a probability space with N i.i.d. coins.
- Let $\underline{u}, \underline{d} > 0$, $S_0 > 0$, and define $S_{n+1} = \begin{cases} \underline{u}S_n & \omega_{n+1} = 1, \\ \underline{d}S_n & \omega_{n+1} = -1. \end{cases}$
- u and d are called the up and down factors respectively.
- Without loss, can assume $\underline{d} < u$.
- Always assume no coins are deterministic: $\underline{p}_1 = \underline{P}(\omega_n = 1) > 0$ and $\underline{q}_1 = 1 - p_1 = \underline{P}(\omega_n = -1) > 0$.
- We have access to a bank with interest rate $\underline{r} > -1$.
- $\underline{D}_n = (1 + r)^{-n}$ be the discount factor (\$1 at time n is worth $\$D_n$ at time 0.)

Theorem 6.1. There exists a (unique) equivalent measure \tilde{P} under which process $D_n S_n$ is a martingale if and only if $d < 1 + r < u$. In this case \tilde{P} is the probability measure obtained by tossing N i.i.d. coins with

$$\tilde{P}(\omega_n = 1) = \tilde{p}_1 = \frac{1 + r - d}{u - d}, \quad \tilde{P}(\omega_n = -1) = \tilde{q}_1 = \frac{u - (1 + r)}{u - d}.$$

Definition 6.2. An equivalent measure \tilde{P} under which $D_n S_n$ is a martingale is called the *risk neutral measure*.

Remark 6.3. If there are more than one risky assets, S^1, \dots, S^k , then we require $D_n S_n^1, \dots, D_n S_n^k$ to all be martingales under the risk neutral measure \tilde{P} .

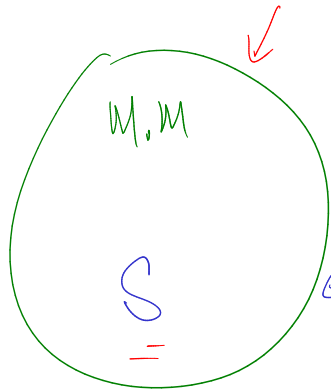
Remark 6.4. The Risk Neutral Pricing Formula says that any security with payoff V_N at time N has arbitrage free price $V_n = \frac{1}{D_n} \tilde{E}_n(D_N V_N)$ at time n .

(Note AFP at time 0 = $V_0 = \frac{1}{1} \tilde{E}(D_N V_N)$)

$V_n = \frac{1}{D_n} \tilde{E}_n(D_N V_N)$

Example 6.5. Consider two markets in the Binomial model setup with the same u, d, r . In the first market the coin flip heads with probability 99%. In the second the coin flips heads with probability 90%. Are the price of call options in these two markets the same?

(assume $d = \frac{1}{u}$)

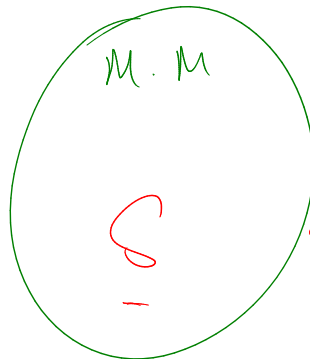


coins flip heads
with prob. 99%

Market 1.

Binomial model r, u, d

$N=2$. $V_2 =$ call option payoff
strike S_0

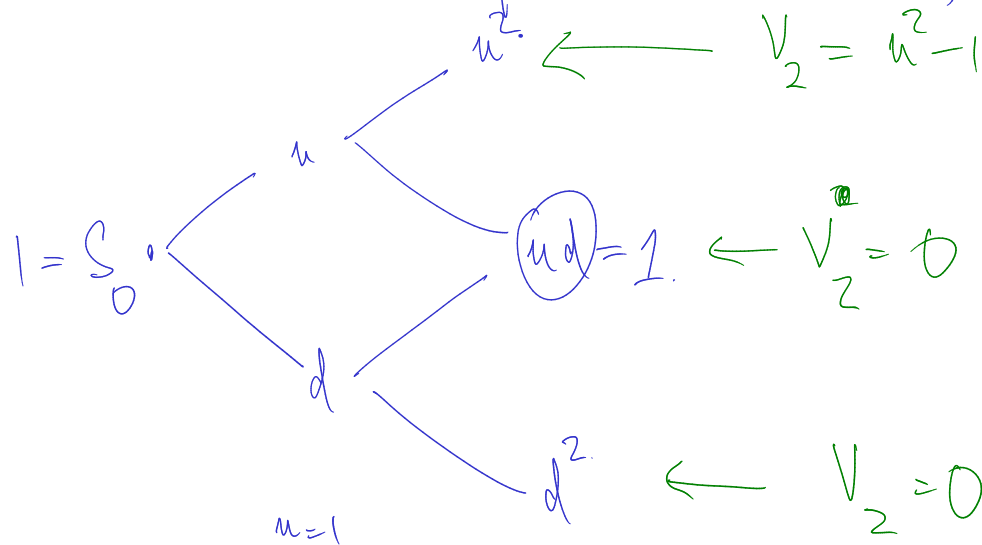


coins flip heads
with prob.
90%

Market 2.

Binomial model r, u, d

Say $S_0 = 1$ & compute V_0 (blue stroke)



$$V_0 = \tilde{E}(D_2 V_2)$$

$$= \frac{(\tilde{\phi}_1)^2 (u^2 - 1)}{(1+r)^2}$$

$$V_0 = \frac{1}{(1+r)^2} \left(\frac{1+r-d}{u-d} \right)^2 (u^2 - 1)$$

Calculation for red stock: Exactly the same!

$$V_0 = \frac{1}{(1+r)^2} \left(\frac{1+r-d}{u-d} \right)^2 (u^2 - 1)$$

- Consider an investor that starts with X_0 wealth, which he divides between cash and the stock.
- If he has Δ_0 shares of stock at time 0, then $X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$.
- We allow the investor to trade at time 1 and hold Δ_1 shares.
- Δ_1 may be random, but must be \mathcal{F}_1 -measurable.
- Continuing further, we see $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$.
- Both X and Δ are adapted processes.

Need Δ_n to be \mathcal{F}_n meas!

Definition 6.6. A self-financing portfolio is a portfolio whose wealth evolves according to

$$\rightarrow X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n),$$

for some adapted process Δ_n .

(self financing means no external cash flow)

Theorem 6.7. Let $d < 1+r < u$, and $\tilde{\mathbf{P}}$ be the risk neutral measure, and X_n represent the wealth of a portfolio at time n . The portfolio is self-financing portfolio if and only if the discounted wealth $D_n X_n$ is a martingale under $\tilde{\mathbf{P}}$.

Remark 6.8. The only thing we will use in this proof is that $D_n S_n$ is a martingale under $\tilde{\mathbf{P}}$. The interest rate r can be a random adapted process. It is also not special to the binomial model – it works for any model for which there is a risk neutral measure.

(Def of $\tilde{\mathbf{P}}$ is that $D_n S_n$ is a $\tilde{\mathbf{P}}$ mg)

IOU

a proof of Thm 6.7.

Before proving Theorem 6.7, we consider a few consequences:

Theorem 6.9. *The multi-period binomial model is arbitrage free if and only if $d < 1 + r < u$.*

Remark 6.10. The first fundamental theorem of asset pricing states that a risk neutral measure exists if and only if the market is arbitrage free. (We will prove this in more generality later.)

Pf: No arbitrage: We say there is no arbitrage in the market if for any self financing portfolio with wealth process X_n ,

we have : If $X_0 = 0$, & $X_N \geq 0$ then $X_N = 0$ almost surely

① Clearly if $1+r \leq d$ or $1+r \geq u \rightarrow$ there is arb.
(You check)

(2) Say $d < 1+r < u$.

NTS there is no arbitrage in the market.

Pf: NTS If $X_0 = 0$, X_n = wealth at time n
of a self financing portfolio

$X_n \geq 0$ then $X_n = 0$

Know by thm 6.7 that If X_n is self financing

then $D_N X_N$ is a \tilde{P} mg.

$$\Rightarrow \underbrace{D_0 X_0}_{\substack{4 \\ 0}} = \tilde{E}_{\text{mg}}(D_N X_N) \quad (\text{mg})$$

$$\Rightarrow \tilde{E}(D_N X_N) = 0 \quad \text{Know } X_N \geq 0 \quad \underline{\text{a.s.}} \\ (D_N > 0).$$

$$\Rightarrow D_N X_N = 0 \quad \text{a.s.} \Rightarrow X_N = 0 \quad \text{a.s.}$$

Theorem 6.11 (Risk Neutral Pricing Formula). *Let $d < 1 + r < u$, and V_N be an \mathcal{F}_N measurable random variable. Consider a security that pays V_N at maturity time N . For any $n \leq N$, the arbitrage free price of this security is given by*

$$V_n = \frac{1}{D_n} \tilde{\mathbf{E}}_n(D_N V_N).$$

Remark 6.12. The replicating strategy can be found by backward induction. Let $\omega = (\omega', \omega_{n+1}, \omega'')$. Then

$$\Delta_n(\omega) = \frac{V_{n+1}(\omega', 1, \omega'') - V_{n+1}(\omega', -1, \omega'')}{(u - d)S_n(\omega)} = \frac{V_{n+1}(\omega', 1) - V_{n+1}(\omega', -1)}{(u - d)S_n(\omega)}$$

Proof of Theorem 6.7 part 1. Suppose X_n is the wealth of a self-financing portfolio. Need to show $D_n X_n$ is a martingale under \tilde{P} .

Proof of Theorem 6.7 part 2. Suppose $D_n X_n$ is a martingale under $\tilde{\mathbf{P}}$. Need to show X_n is the wealth of a self-financing portfolio.

6.2. State processes.

Question 6.13. *Consider the N -period binomial model, and a security with payoff V_N . Let X_n be the arbitrage free price at time $n \leq N$, and Δ_n be the number of shares in the replicating portfolio. What is an algorithm to find X_n, Δ_n for all $n \leq N$? How much is the computational time?*

Theorem 6.14. Suppose a security pays $V_N = g(S_N)$ at maturity N for some (non-random) function g . Then the arbitrage free price at time $n \leq N$ is given by $V_n = f_n(S_n)$, where:

$$(1) f_N(x) = V_N(x) \text{ for } x \in \text{Range}(S_N).$$

$$(2) f_n(x) = \frac{1}{1+r}(\tilde{p}f_{n+1}(ux) + \tilde{q}f_{n+1}(dx)) \text{ for } x \in \text{Range}(S_n).$$

Remark 6.15. Reduces the computational time from $O(2^N)$ to $O(\sum_0^N |\text{Range}(S_n)|) = O(N^2)$ for the Binomial model.

Remark 6.16. Can solve this to get $f_n(x) = \frac{1}{(1+r)^{N-n}} \sum_{k=0}^{N-n} \binom{N-n}{k} \tilde{p}^k \tilde{q}^{N-n-k} f_N(xu^k d^{N-n-k})$

Question 6.17. *How do we handle other securities? E.g. Asian options (of the form $g(\sum_0^N S_k)$)?*