Lecture 12 (9/27). Please enable your video if you can.

6. The multi-period binomial model

6.1. Risk Neutral Pricing.

- In the multi-period binomial model we assume $\Omega = \{\pm 1\}^N$ corresponds to a probability space with N i.i.d. coins.
- Let $u, d > 0$, $S_0 > 0$, and define $S_{n+1} = \begin{cases} uS_n & \omega_{n+1} = 1\,, \\ dS_n & \omega_{n+1} = -1\,. \end{cases}$
- u and d are called the up and down factors respectively.
- Without loss, can assume $d < u$.
- Always assume no coins are deterministic: $p_1 = \boldsymbol{P}(\omega_n = 1) > 0$ and $q_1 = 1 - p_1 = \boldsymbol{P}(\omega_n = -1) > 0$.
- We have access to a bank with interest rate $r > -1$.
- $D_n = (1+r)^{-n}$ be the discount factor (\$1 at time n is worth \$$D_n$ at time 0.)

Theorem 6.1. *There exists a (unique) equivalent measure $\tilde{\boldsymbol{P}}$ under which process $D_n S_n$ is a martingale* if and only if $d < 1 + r < u$. *In this case $\tilde{\boldsymbol{P}}$ is the probability measure obtained by tossing N i.i.d. coins with*

$$\tilde{\boldsymbol{P}}(\omega_n = 1) = \tilde{p}_1 = \boxed{\frac{1+r-d}{u-d}}\,, \qquad \tilde{\boldsymbol{P}}(\omega_n = -1) = \tilde{q}_1 = \boxed{\frac{u-(1+r)}{u-d}}\,.$$

Definition 6.2. An equivalent measure $\tilde{\boldsymbol{P}}$ under which $D_n S_n$ is a martingale is called the *risk neutral measure.*

Remark 6.3. If there are more than one risky assets, $S^1, \dots, S^k$, then we require $D_n S_n^1, \dots, D_n S_n^k$ to all be martingales under the risk neutral measure $\tilde{\boldsymbol{P}}$.

Remark 6.4. The *Risk Neutral Pricing Formula* says that any security with payoff V_N at time N has arbitrage free price $\boxed{V_n = \frac{1}{D_n}\tilde{\boldsymbol{E}}_n(D_N V_N)}$ at time n.

(Note AFP at time $0 = V_0 = \frac{1}{1}\tilde{E}(D_N V_N)$)

Example 6.5. Consider two markets in the Binomial model setup with the same u, d, r. In the first market the coin flip heads with probability 99%. In the second the coin flips heads with probability 90%. Are the price of call options in these two markets the same?

(assume $d = \frac{1}{u}$)

M.M

S

coins flip heads with prob. 99%

Market 1.

Binomial model r, u, d

$N = 2$. V_2 = call option payoff strike S_0

M.M

S

coins flip heads with prob 90%

Market 2.

Binomial model r, u, d

Say $S_0 = 1$ & compute V_0 (blue stock)

$$
\begin{tikzcd}[column sep=small]
 & & u^2 & V_2 = u^2 - 1 \arrow[l] \\
 & u \arrow[ur] \arrow[dr] & & \\
1 = S_0 \arrow[ur] \arrow[dr] & & ud = 1 & V_2 = 0 \arrow[l] \\
 & d \arrow[ur] \arrow[dr] & & \\
 & & d^2 & V_2 = 0 \arrow[l]
\end{tikzcd}
$$

$u=1$

$$V_0 = \tilde{E}(D_2 V_2)$$

$$= \frac{(\tilde{p}_1)^2 (u^2-1)}{(1+r)^2}$$

$$\boxed{V_0 = \frac{1}{(1+r)^2}\left(\frac{1+r-d}{u-d}\right)^2 (u^2-1)}$$

Calculation for red stock: Exactly the same!

$$V_0 = \boxed{\frac{1}{(1+r)^2}\left(\frac{1+r-d}{u-d}\right)^2(u^2-1)}$$

- Consider an investor that starts with $\boxed{X_0}$ wealth, which he divides between cash and the stock.
- If he has Δ_0 shares of stock at time 0, then $X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$.
- We allow the investor to trade at time 1 and hold Δ_1 shares.
- Δ_1 may be random, *but must be $\mathcal{F}_1$-measurable.*
- Continuing further, we see $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$.
- *Both X and Δ are adapted processes.*

Need Δ_n to be $\mathcal{F}_n$ meas!

(self financing means no external cash flow)

Definition 6.6. A *self-financing portfolio* is a portfolio whose wealth evolves according to

$$\longrightarrow \quad X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)\,,$$

for some adapted process Δ_n.

Theorem 6.7. *Let $d < 1 + r < u$, and $\tilde{\boldsymbol{P}}$ be the risk neutral measure, and X_n represent the wealth of a portfolio at time n. The portfolio is self-financing portfolio if and only if the discounted wealth $D_n X_n$ is a martingale under $\tilde{\boldsymbol{P}}$.*

Remark 6.8. The only thing we will use in this proof is that $D_n S_n$ is a martingale under $\tilde{\boldsymbol{P}}$. The interest rate r can be a *random* adapted process. It is also not special to the binomial model – it works for any model for which there is a risk neutral measure.

(Def of $\tilde{P}$ is that $D_n S_n$ is a $\tilde{P}$ mg)

IOU

a proof of Thm 6.7.

Before proving Theorem 6.7, we consider a few consequences:

Theorem 6.9. *The multi-period binomial model is arbitrage free if and only if* $\boxed{d < 1 + r < u.}$

Remark 6.10. The first fundamental theorem of asset pricing states that a risk neutral measure exists if and only if the market is arbitrage free. (We will prove this in more generality later.)

Pf: No arbitrage: We say there is no arbitrage in the market if for any self financing portfolio with wealth process X_n, we have: If $X_0 = 0$, & $\underbrace{X_N \geq 0}$ then $X_N = 0$ almost surely

(1) Clearly the $1+r \leq d$ or $1+r \geq u \longrightarrow$ there is arb. (You check)

(2) Say $d < 1+r < u$.

NTS there is no arb in the market.

Pf: NTS If $X_0 = 0$, X_n = wealth at time n of a self financing portfolio

$X_N \geq 0$ then $X_N = 0$

Know by thm 6.7 that If X_n is self fining then $D_n X_n$ is a $\tilde{P}$ mg.

$$\Rightarrow \underbrace{D_0 X_0}_{\substack{\| \\ 0}} = \tilde{E}\left(D_N X_N\right) \qquad (\because \text{mg})$$

$$\Rightarrow \tilde{E}\left(\underbrace{D_N X_N}\right) = 0 \qquad \text{Know } X_N \geq 0 \ \text{a.s.} \ (D_N > 0).$$

$$\Rightarrow D_N X_N = 0 \ \text{a.s.} \Rightarrow X_N = 0 \ \text{a.s.}$$

Theorem 6.11 (Risk Neutral Pricing Formula). *Let $d < 1 + r < u$, and V_N be an $\mathcal{F}_N$ measurable random variable. Consider a security that pays V_N at maturity time N. For any $n \leqslant N$, the arbitrage free price of this security is given by*

$$V_n = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n(D_N V_N) .$$

Remark 6.12. The replicating strategy can be found by backward induction. Let $\omega = (\omega', \omega_{n+1}, \omega'')$. Then

$$\Delta_n(\omega) = \frac{V_{n+1}(\omega', 1, \omega'') - V_{n+1}(\omega', -1, \omega'')}{(u-d)S_n(\omega)} = \frac{V_{n+1}(\omega', 1) - V_{n+1}(\omega', -1)}{(u-d)S_n(\omega)}$$

Proof of Theorem 6.7 part 1. Suppose X_n is the wealth of a self-financing portfolio. Need to show $D_n X_n$ is a martingale under $\tilde{\boldsymbol{P}}$.

Proof of Theorem 6.7 part 2. Suppose $D_n X_n$ is a martingale under $\tilde{\boldsymbol{P}}$. Need to show X_n is the wealth of a self-financing portfolio.

6.2. **State processes.**

Question 6.13. *Consider the N-period binomial model, and a security with payoff V_N. Let X_n be the arbitrage free price at time $n \leqslant N$, and Δ_n be the number of shares in the replicating portfolio. What is an algorithm to find X_n, Δ_n for all $n \leqslant N$? How much is the computational time?*

Theorem 6.14. *Suppose a security pays $V_N = g(S_N)$ at maturity N for some (non-random) function g. Then the arbitrage free price at time $n \leqslant N$ is given by $V_n = f_n(S_n)$, where:*

(1) $f_N(x) = V_N(x)$ *for* $x \in \text{Range}(S_N)$.

(2) $f_n(x) = \dfrac{1}{1+r}(\tilde{p} f_{n+1}(ux) + \tilde{q} f_{n+1}(dx))$ *for* $x \in \text{Range}(S_n)$.

Remark 6.15. Reduces the computational time from $O(2^N)$ to $O(\sum_0^N |\text{Range}(S_n)|) = O(N^2)$ for the Binomial model.

Remark 6.16. Can solve this to get $f_n(x) = \dfrac{1}{(1+r)^{N-n}} \sum_{k=0}^{N-n} \binom{N-n}{k} \tilde{p}^k \tilde{q}^{N-n-k} f_N(xu^k d^{N-n-k})$

Question 6.17. *How do we handle other securities? E.g. Asian options (of the form $g(\sum_0^N S_k)$)?*