Indep : ① Events : A & B are indep if $\overline{P(A \cap B) = P(A)P(B)}$

② R.V. : X & Y are indep if $\forall x \in \text{Range}(X)$, $y \in \text{Range}(Y)$ $\overline{P(X=x, Y=y) = P(X=x) \cdot P(Y=y)}$

③ R.V. & "σ-alg"

Collection of events

We say X is ind of $\mathcal{F}_n$ if

$\forall\, x \in \cancel{\mathbb{R}}$ Range(X), & every event $A \in \mathcal{G}_n$,

the events $\{X = x\}$ & A are indep

(i.e. $P(\{X = x\} \cap A) = P(X = x)\, P(A)$).

Eg: $X_{n+1} = X_n + \omega_{n+1} + a$ $\qquad \omega_n \in \{\pm 1\}$

(iid fair coins).

Want $\tilde{E}_n(X_{n+1}) = X_n$

Want $\tilde{E}_n(X_n + \omega_{n+1} + a) = X_n$.

$\iff X_n + a + \tilde{E}_n \omega_{n+1} = X_n$ $\quad$ ($\because$ X_n is $\mathcal{F}_n$ meas)

$\iff \quad \tilde{E}_n \omega_{n+1} = -a.$

Choose $\tilde{P}$ so that all coins are still $\underline{\text{iid}}$ & land heads with prob $\tilde{p}_1$ & tails with prob $\tilde{q}_1$.

$\Rightarrow \tilde{E}_n \omega_{n+1} = \tilde{E}\omega_{n+1}$

$$\iff \underbrace{\tilde{p}_1 \cdot 1 + \tilde{q}_1 \cdot (-1)}_{\text{looks like } \tilde{E}\omega_{n+1} \text{ not } \tilde{E}_n \omega_{n+1}} = -a$$

Solve for $\tilde{p}_1$

Note A & B are ind under P

then $P(A \cap B) = P(A)\, P(B)$

$\not\Rightarrow$ A & B are indep under $\tilde{P}$

HW 4 Q2

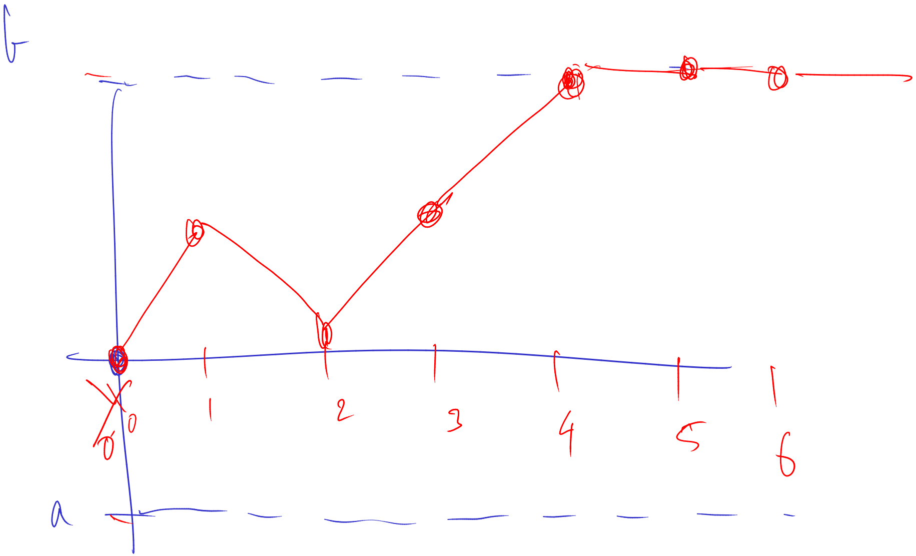
X_n iid , ± 1

$Y_{n+1} = Y_n + X_{n+1}$ if $Y_n \in (a, b)$

$Y_{n+1} = Y_n$ otherwise.

1a

$$E_n Y_{n+1} = Y_n$$

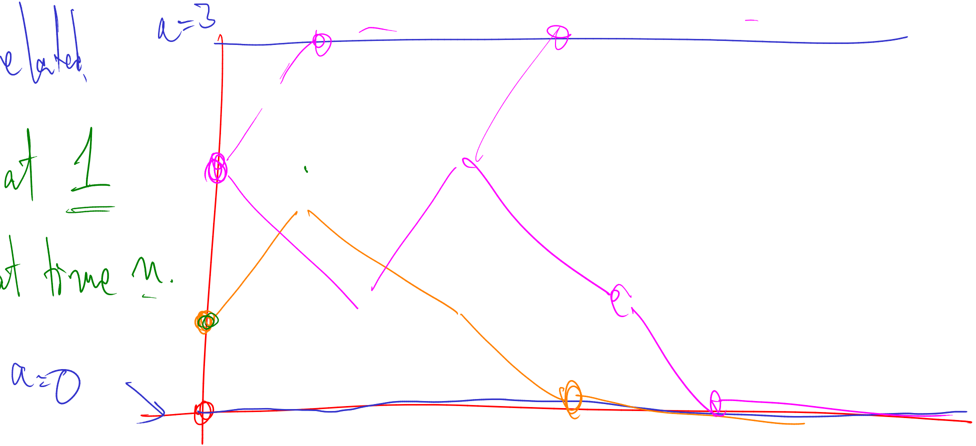


2b) $a=0$ $b=3$. $p_n = P(Y_n = 0)$ if $Y_0 = 1$

& $q_n = P(Y_n = 0)$ if $Y_0 = 2$.

p_n & p_{n+2} are related.

p_n = prob that that you start at 1 & and are stuck at 0 at time n.



At time n $\longrightarrow$ not stuck to 0 & start at 1.

$\Rightarrow$ 1st step was upward.

$$\underbrace{P(Y_n \neq 0 \,\&\, Y_0 = 1)}_{1 - p_n} = \frac{1}{2} \underbrace{P(Y_n \neq 0 \,\&\, Y_1 = 2)}_{1 - q_{n-1}}$$

Lecture

V_N = pay off at N of a security

(Eg Call option : $V_N = (S_N - K)^+$)

V_n = AFP at time $n \leq N$

$$= \frac{1}{D_n} \tilde{E}_n (D_N V_N)$$

Q3] $u = 2$, $d = \frac{1}{2}$, $r = \frac{1}{4}$

Find AFP at time 0 & time 1 of a call option with strike 10 & $\boxed{N = 2.}$

$$V_N = (S_N - 10)^+ \quad (N = 2)$$

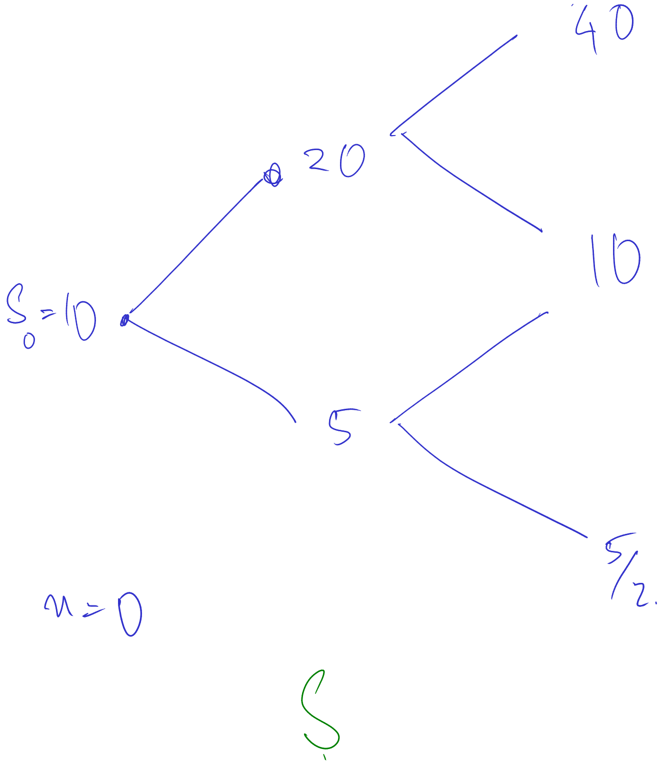
$V_2 = (S_2 - 10)^+$. Find V_0 & V_1

$$\boxed{V_1 = \frac{1}{D_1} \tilde{E}_1 D_2 (S_2 - 10)^+}$$

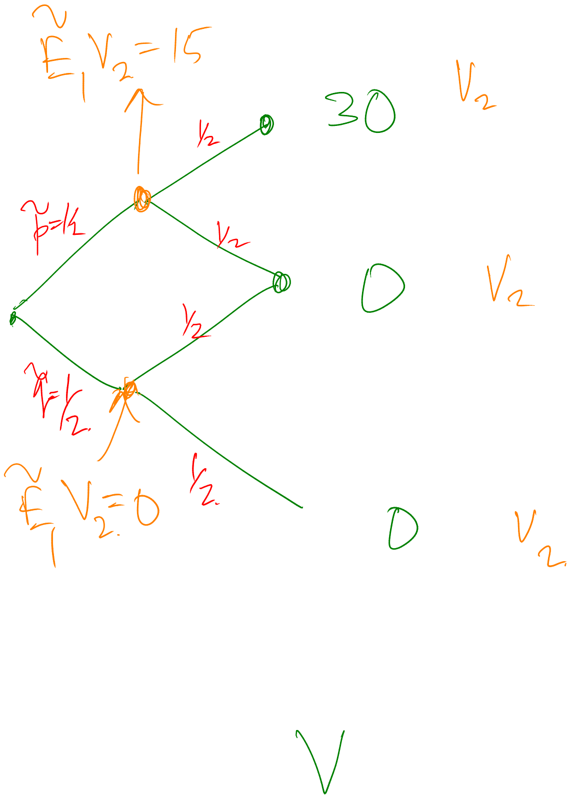
$$V_0 = \frac{1}{D_0}\tilde{E}_0\left(S_2 - 10\right)^+ = \frac{1}{D_0}\tilde{E}\left(S_2 - 10\right)^+.$$

find $\tilde{p}$:
$$\tilde{p} = \frac{1+r-d}{u-d} = \frac{1+\frac{1}{4}-\frac{1}{2}}{2-\frac{1}{2}} = \frac{\left(\frac{5-2}{4}\right)}{3/2} = \frac{1}{2}.$$

$$\tilde{q} = \frac{1}{2}.$$



$S_0 = 10$
20
40
10
5
5/2.
n = 0
S



$\tilde{E}_1 V_2 = 15$
30
V_2
0
V_2
0
V_2
$\tilde{p} = 1/2$
$\tilde{q} = 1/2.$
1/2
1/2
1/2
1/2.
$\tilde{E}_1 V_2 = 0$
V

$$V_1 = \frac{1}{D_1}\tilde{E}_1 D_2 V_2 = (1+r)\tilde{E}_1 V_2.$$

$$V_1 = \begin{cases} \frac{5}{4}\cdot 15 & \text{if } \omega_1 = 1 \\ 0 & \text{if } \omega_1 = -1 \end{cases}$$