Lecture 11 (9/24). Please Enable Your Video If you Can

Last time: ① Change of measure (Equivalent Measures)

② Risk Neutral Pricing formula.

5.5. **Change of measure.**

↙ Last time.

Example 5.52. Consider i.i.d. coin tosses with $\boldsymbol{P}(\omega_n = 1) = p_1$ and $\boldsymbol{P}(\omega_n = -1) = q_1 = 1 - p_1$. Let $u, d > 0$, $r > -1$. Let $S_{n+1}(\omega) = uS_n(\omega)$ if $\omega_{n+1} = 1$, and $S_{n+1}(\omega) = dS_n(\omega)$ if $\omega_{n+1} = -1$. Let $\boxed{D_n = (1+r)^{-n}}$ be the "discount factor".

Suppose we now invented a new "risk neutral" coin that comes up heads with probability $\tilde{p}_1$ and tails with probability $\tilde{q}_1 = 1 - \tilde{p}_1$. Let $\tilde{\boldsymbol{P}}, \tilde{\boldsymbol{E}}_n$ etc. denote the probability and conditional expectation with respect to the new "risk neutral" coin. Find $\tilde{p}_1$ so that $D_n S_n$ is a $\tilde{\boldsymbol{P}}$ martingale.

Theorem 5.53. *Consider a market where S_n above models a stock price, and r is the interest rate with $0 < d < 1 + r < u$. The coins land heads and tails with probability p_1 and q_1 respectively. If you have a derivative security that pays V_N at time N, then the arbitrage free price of this security at time $n \leqslant N$ is given by*

$$\boxed{V_n = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n D_N V_N = (1+r)^{n-N} \tilde{\boldsymbol{E}}_n V_N \,.}$$

(IOU Pf.)

Remark 5.54. Even though the stock price changes according to a coin that flips heads with probability p_1, the arbitrage free price is computed using conditional expectations using the *risk neutral probability*. So when computing $\tilde{\boldsymbol{E}}_n V_N$, we use our new invented "risk neutral" coin that flips heads with probability $\tilde{p}_1$ and tails with probability $\tilde{q}_1$.

Cond exp w.r.t the risk neutral measure $\tilde{P}$

(not P!!)

- Let $p\colon \Omega \to [0,1]$ be a probability mass function on Ω, and $\boldsymbol{P}(A) = \sum_{\omega \in A} p(\omega)$ be the probability measure. ✓ last time.
- Let $\tilde{p}\colon \Omega \to [0,1]$ be another probability mass function, and define a second probability measure $\tilde{\boldsymbol{P}}$ by $\tilde{\boldsymbol{P}}(A) = \sum_{\omega \in A} \tilde{p}(\omega)$.

Definition 5.55. We say $\boldsymbol{P}$ and $\tilde{\boldsymbol{P}}$ are equivalent if for every $A \in \mathcal{F}_N$, $\boldsymbol{P}(A) = 0$ if and only if $\tilde{\boldsymbol{P}}(A) = 0$.

Remark 5.56. When Ω is finite, $\boldsymbol{P}$ and $\tilde{\boldsymbol{P}}$ are equivalent if and only if we have $p(\omega) = 0 \iff \tilde{p}(\omega) = 0$ for all $\omega \in \Omega$.

We let $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{E}}_n$ denote the expectation and conditional expectations with respect to $\tilde{\boldsymbol{P}}$ respectively.

$$P(A) = \sum_{\omega \in A} p(\omega) \quad (\text{prob of } A \text{ occuring under } P)$$

$$\tilde{P}(A) = \sum_{\omega \in A} \tilde{p}(\omega) \quad (\quad " \quad " \quad " \quad " \quad " \quad \tilde{P})$$

Work out Example 5.52. Find $\tilde{p}_1$ & $\tilde{q}_1$ so that $D_n S_n$ is a $\tilde{P}$ mg.

$$X_{n+1} = \begin{cases} u & \text{if } n+1^{\text{th}} \text{ coin is heads} \\ d & \text{" " " " tails.} \end{cases}$$

$S_{n+1} = X_{n+1} S_n$ & X_{n+1} is ind of $\mathcal{F}_n$. (under $\tilde{P}$).

$$\tilde{E}_n S_{n+1} = \tilde{E}_n \left(X_{n+1} S_n \right) = S_n \tilde{E}_n X_{n+1} = S_n \tilde{E} X_{n+1}$$

$$= \underbrace{S_n \left(\tilde{p}_1 u + \tilde{q}_1 d \right)}$$

Want $D_n S_n$ to be a $\tilde{P}$ mg

$$\iff \tilde{E}_n \left(D_{n+1} S_{n+1} \right) \overset{\text{Want}}{=} D_n S_n.$$

$$\iff \frac{1}{(1+r)^{n+1}} \tilde{E}_n S_{n+1} \overset{\text{Want}}{=} \frac{1}{(1+r)^n} S_n \iff \boxed{\tilde{E}_n S_{n+1} \overset{\text{Want}}{=} (1+r) S_n}$$

Use $\tilde{E}_n S_{n+1} = (\tilde{p}_1 u + \tilde{q}_1 d) S_n.$

$$\Rightarrow (\tilde{p}_1 u + \tilde{q}_1 d)\cancel{S_n} = (1+r)\cancel{S_n}.$$

$$\Rightarrow \boxed{\tilde{p}_1 u + \tilde{q}_1 d = 1+r}$$

$$\Rightarrow \tilde{p}_1 u + (1-\tilde{p}_1) d = 1+r \iff \boxed{\tilde{p}_1 = \frac{1+r-d}{u-d}}$$

$\therefore$ If we chose $\tilde{p}_1 = \dfrac{1+r-d}{u-d}$ & $\tilde{q}_1 = 1-\tilde{p}_1 = \dfrac{u-(1+r)}{u-d}$

then $D_n S_n$ is a $\tilde{P}$ mg.

Example 5.57. Let Ω be the sample space corresponding to N i.i.d. fair coins (heads is 1, tails is -1). Let $a \in \mathbb{R}$ and define $X_{n+1}(\omega) = X_n(\omega) + \omega_{n+1} + a$. For what a is there an equivalent measure $\tilde{\boldsymbol{P}}$ such that X is a martingale?

(Try & work this out)

6. The multi-period binomial model

6.1. Risk Neutral Pricing.

- In the multi-period binomial model we assume $\Omega = \{\pm 1\}^N$ corresponds to a probability space with N i.i.d. coins.
- Let $u, d > 0$, $S_0 > 0$, and define $S_{n+1} = \begin{cases} uS_n & \omega_{n+1} = 1\,, \\ dS_n & \omega_{n+1} = -1\,. \end{cases}$
- u and d are called the up and down factors respectively.
- Without loss, can assume $d < u$.
- Always assume no coins are deterministic: $p_1 = \boldsymbol{P}(\omega_n = 1) > 0$ and $q_1 = 1 - p_1 = \boldsymbol{P}(\omega_n = -1) > 0$.
- We have access to a bank with interest rate $r > -1$.
- $D_n = (1+r)^{-n}$ be the discount factor (\$1 at time n is worth \$$D_n$ at time 0.)

Theorem 6.1. *There exists a (unique) equivalent measure $\tilde{\boldsymbol{P}}$ under which process $D_n S_n$ is a martingale* if and only if $\boxed{d < 1 + r < u}$. *In this case $\tilde{\boldsymbol{P}}$ is the probability measure obtained by tossing N i.i.d. coins with*

$$\tilde{\boldsymbol{P}}(\omega_n = 1) = \tilde{p}_1 = \frac{1+r-d}{u-d}\,, \qquad \tilde{\boldsymbol{P}}(\omega_n = -1) = \tilde{q}_1 = \frac{u-(1+r)}{u-d}\,.$$

Definition 6.2. An equivalent measure $\tilde{\boldsymbol{P}}$ under which $D_n S_n$ is a martingale is called the *risk neutral measure.*

Remark 6.3. If there are more than one risky assets, S^1, ..., S^k, then we require $D_n S^1_n$, ..., $D_n S^k_n$ to all be martingales under the risk neutral measure $\tilde{\boldsymbol{P}}$.

Remark 6.4. The *Risk Neutral Pricing Formula* says that any security with payoff V_N at time N has arbitrage free price $\boxed{V_n = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n(D_N V_N)}$ at time n.

Pf of thm 6.1:

1 If $d < 1+r < u$, then chose $\tilde{p}_1 = \dfrac{1+r-d}{u-d} \in (0,1)$

$\tilde{q}_1 = \dfrac{u-(1+r)}{u-d} \in (0,1)$

By example above we know $\tilde{E}_n(D_{n+1} S_{n+1}) = D_n S_n$

$\Rightarrow \tilde{P}$ can be obtained by using iid tosses of a coin that has heads with pr $\tilde{p}_1$ & tails with pr $\tilde{q}_1$.

m

(2) Reverse direction:

<u>Only</u> choice of $\tilde{p}_1$ & $\tilde{q}_1$ for which $\tilde{E}_n(D_{n+1}S_{n+1}) = D_n S_n$

is given by $\tilde{p}_1 = \dfrac{1+r-d}{u-d}$ & $\tilde{q}_1 = \dfrac{u-(1+r)}{u-d}$.

If $1+r < d \Rightarrow \tilde{p}_1 < 0 \Rightarrow \tilde{P}$ is not a prob measure.

If $1+r = d \Rightarrow \tilde{p}_1 = 0$. $\tilde{P}$ is a prob meas

but $\tilde{P}$ is NOT equiv to P.

If $1+r > u \Rightarrow \tilde{q}_1 < 0 \Rightarrow \tilde{P}$ is not a prob meas.

& $1+r = u \Rightarrow \tilde{q}_1 = 0 \Rightarrow \tilde{P}$ is a prob meas

but NOT equiv to P.

QED.