Lecture 10 (9/22). Please enable your video if you can.

Last time: Martingales $\longrightarrow$ Fair game: $\boxed{E_n M_{n+1} = M_n.}$

(All processes are adapted)

Last time: M is a mg $\Rightarrow$ $EM_n = EM_1$ $\forall n$.

(constant expectation)

Converse is false: $EM_n = EM_1$ $\forall M_{n+1}$ $\not\Rightarrow$ M is a mg.

Question 5.50. *Let* ξ_n *be a martingale with* $\boxed{\boldsymbol{E}\xi_1 = 0}$. *Let* Δ_n *be an adapted process,* $X_0 \in \mathbb{R}$ *and define* $X_{n+1} = X_n + \Delta_n \xi_{n+1}$. *Is* X *a martingale?* (Need not be)

$\leftarrow E\xi_n = 0 \ \forall n.$

Remark 5.51. Think of ξ_n as the outcome of a fair game being played. You decide to bet on this game. Let Δ_n be your bet at time n; your return from this bet is $\Delta_n \xi_{n+1}$, and thus your cumulative return at time $n+1$ is $X_{n+1} = X_n + \Delta_n \xi_{n+1}$.

Pf: ① Adapted: $X_{n+1} = \underbrace{\underbrace{X_n}_{\substack{\mathcal{F}_n\text{-meas} \\ (\text{ind})}} + \underbrace{\Delta_n}_{\substack{\mathcal{F}_n\text{-meas} \\ (\text{assumption})}} \underbrace{\xi_{n+1}}_{\mathcal{F}_{n+1}\text{-meas}}}_{\mathcal{F}_{n+1} \text{ meas}}$ $\implies$ adapted.

② NTS $E_n X_{n+1} = X_n$

Pf: $\mathbb{E}_n X_{n+1} = \mathbb{E}_n \left(X_n + \Delta_n \xi_{n+1} \right)$

$$= \mathbb{E}_n X_n + \mathbb{E}_n \left(\Delta_n \xi_{n+1} \right)$$

$$= X_n + \mathbb{E}_n \left(\Delta_n \xi_{n+1} \right)$$

($\mathbb{E}_n X_n = X_n$ since X_n is $\mathcal{F}_n$-meas)

($\because \Delta_n$ is $\mathcal{F}_n$ meas),

$$= X_n + \Delta_n \mathbb{E}_n \xi_{n+1}$$

$$= X_n + \Delta_n \xi_n$$ $\longleftarrow$ Doesn't simplify further.

(Will work if we knew ξ_{n+1} is ind of $\mathcal{F}_n$. Otherwise can't simplify further.)

<u>Proof:</u> Say M is a mg. Let $\xi_{n+1} = M_{n+1} - M_n$.

$\Delta_n \to$ any adapted process.

$$X_{n+1} = \underbrace{X_n} + \underline{\Delta_n} (M_{n+1} - M_n) = X_n + \Delta_n \xi_{n+1}.$$

Claim: In this case X is a mg.

Pf: $E_n X_{n+1} = E_n \big(X_n + \Delta_n (M_{n+1} - M_n) \big)$

$= \underbrace{X_n} + \Delta_n E_n (M_{n+1} - M_n)$

$= X_n$ $\quad \big(\because E_n M_{n+1} = M_n \Rightarrow E_n (M_{n+1} - M_n) = 0 \big)$

5.5. **Change of measure.**

Example 5.52. Consider i.i.d. coin tosses with $\boldsymbol{P}(\omega_n = 1) = p_1$ and $\boldsymbol{P}(\omega_n = -1) = q_1 = 1 - p_1$. Let $u, d > 0$, $r > -1$. Let $S_{n+1}(\omega) = uS_n(\omega)$ if $\omega_{n+1} = 1$, and $S_{n+1}(\omega) = dS_n(\omega)$ if $\omega_{n+1} = -1$. Let $D_n = (1+r)^{-n}$ be the "discount factor".

Suppose we now invented a new "risk neutral" coin that comes up heads with probability $\tilde{p}_1$ and tails with probability $\tilde{q}_1 = 1 - \tilde{p}_1$. Let $\tilde{\boldsymbol{P}}, \tilde{\boldsymbol{E}}_n$ etc. denote the probability and conditional expectation with respect to the new "risk neutral" coin. Find $\tilde{p}_1$ so that $D_n S_n$ is a $\tilde{\boldsymbol{P}}$ martingale.

Theorem 5.53. *Consider a market where S_n above models a stock price, and r is the interest rate with $0 < d < 1 + r < u$. The coins land heads and tails with probability p_1 and q_1 respectively. If you have a derivative security that pays V_N at time N, then the arbitrage free price of this security at time $n \leqslant N$ is given by*

$$V_n = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n D_N V_N = (1+r)^{n-N} \tilde{\boldsymbol{E}}_n V_N .$$

(IOU a proof → 1 week).

Remark 5.54. Even though the stock price changes according to a coin that flips heads with probability p_1, the arbitrage free price is computed using conditional expectations using the *risk neutral probability*. So when computing $\tilde{\boldsymbol{E}}_n V_N$, we use our new invented "risk neutral" coin that flips heads with probability $\tilde{p}_1$ and tails with probability $\tilde{q}_1$.

- Let $p\colon \Omega \to [0,1]$ be a probability mass function on Ω, and $\boldsymbol{P}(A) = \sum_{\omega \in A} p(\omega)$ be the probability measure.
- Let $\tilde{p}\colon \Omega \to [0,1]$ be another probability mass function, and define a second probability measure $\tilde{\boldsymbol{P}}$ by $\tilde{\boldsymbol{P}}(A) = \sum_{\omega \in A} \tilde{p}(\omega)$.

Definition 5.55. We say $\boldsymbol{P}$ and $\tilde{\boldsymbol{P}}$ are equivalent if for every $A \in \mathcal{F}_N$, $\boldsymbol{P}(A) = 0$ if and only if $\tilde{\boldsymbol{P}}(A) = 0$.

Remark 5.56. When Ω is finite, $\boldsymbol{P}$ and $\tilde{\boldsymbol{P}}$ are equivalent if and only if we have $\boxed{p(\omega) = 0} \iff \boxed{\tilde{p}(\omega) = 0}$ for all $\omega \in \Omega$.

We let $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{E}}_n$ denote the expectation and conditional expectations with respect to $\tilde{\boldsymbol{P}}$ respectively.

$$\tilde{E} X = \sum X(\omega)\, \tilde{p}(\omega)$$

$$E X = \sum X(\omega)\, p(\omega)$$

Work out Example 5.52

Find $\tilde{p}_1$

Know

$$S_n \begin{array}{l} \xrightarrow{p_1} uS_n \\ \xrightarrow{q_1} dS_n \end{array} \qquad \text{(old coin)}$$

$$S_n \begin{array}{l} \xrightarrow{\tilde{p}_1} uS_n \\ \xrightarrow{\tilde{q}_1} dS_n \end{array} \qquad \text{(new coin)}$$

Goal: Want $\tilde{\mathbb{P}}$ so that $D_n S_n$ is a $\tilde{\mathbb{P}}$ mg.

i.e. $\tilde{\mathbb{E}}_n(D_{n+1} S_{n+1}) = D_n S_n$.

Let's compute $\tilde{\mathbb{E}}_n S_{n+1}$:

Let $X_{n+1}(\omega) = \begin{cases} u & \omega_{n+1} = 1 \\ d & \omega_{n+1} = -1 \end{cases}$

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Note $S_{n+1} = X_{n+1} S_n$ & X_{n+1} is $\mathcal{F}_{n+1}$ meas.

& X_{n+1} is ind of $\mathcal{F}_n$!

$$\Rightarrow \tilde{E}_n S_{n+1} = \tilde{E}_n \left(X_{n+1} S_n \right)$$

$$= S_n \tilde{E}_n X_{n+1} \qquad (\because S_n \text{ is } \mathcal{F}_n \text{ meas})$$

$$= S_n \left(\tilde{E} X_{n+1} \right) \qquad (\because X_{n+1} \text{ is ind of } \mathcal{F}_n)$$

$$= \underbrace{S_n\left(\tilde{p}_1 u + (1-\tilde{p}_1) d\right)}.$$

$$\Rightarrow \tilde{E}_n S_{n+1} = \left(\tilde{p}_1 u + (1-\tilde{p}_1) d\right) \cdot S_n .$$

Want $D_n S_n$ to be a $\tilde{P}$ mg.

Want $\tilde{E}_n\left(D_{n+1} S_{n+1}\right) = D_n S_n.$

$$\Leftrightarrow \tilde{E}_n\left((1+r)^{-(n+1)} S_{n+1}\right) = (1+r)^{-n} S_n.$$

$$\Leftrightarrow \quad \tilde{E}_n S_{n+1} = (1+r) S_n.$$

$$\Leftrightarrow \left(\tilde{p}_1 u + (1-\tilde{p}_1) d\right) S_n = (1+r) S_n$$

$$\Leftrightarrow \boxed{\tilde{p}_1 u + (1-\tilde{p}_1) d = 1+r} \rightarrow$$

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