HW 3 Q5

$$\sum_{i=1}^{m} \sum_{\omega \in A \cap \{X = x_i\}} f(x_i, Y(\omega))\, p(\omega)$$

$X \to$ ind of $\mathcal{F}_n$

$Y \to \mathcal{F}_n$ meas.

$\Rightarrow$ Want $\underbrace{E_n f(X, Y)}_{} = \underbrace{\sum f(x_i, Y)\, P(X = x_i)}_{}$

NTS (1) RHS is $\mathcal{F}_n$ meas

(2) $\forall A \in \mathcal{F}_n$, $\underbrace{\sum_{\omega \in A} f(x_i, Y(\omega))\, P(X = x_i)\, p(\omega) = \sum_{\omega \in A} f(X(\omega), Y(\omega))\, p(\omega)}_{}$

$E_n Z$ is the unique RV s.t.

① $E_n Z$ is $\mathcal{F}_n$ meas

& ② $\forall A \in \mathcal{F}_n$, $\sum_{\omega \in A} \underbrace{E_n Z}_{W}(\omega)\, p(\omega) = \sum_{\omega \in A} Z(\omega)\, p(\omega)$

Trying to show $E_n Z = \underline{\underline{W}}$

Check $W$ satisfies ① & ②.

$$\sum_{\omega \in A} \underbrace{\cancel{E_n Z(\omega)}}_{W(\omega)} p(\omega) = \sum_{\omega \in A} Z(\omega)\, p(\omega)$$

$\mathrm{Range}(X) = \{x_1 \cdots x_m\}$ $\mathrm{Range}(Y) = \{y_1, \cdots y_n\}$

$$\sum_{\omega \in A} f(X(\omega), Y(\omega))\, p(\omega) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{\omega \in A \cap \{X = x_i, Y = y_j\}} f(x_i, y_j)\, p(\omega)$$

$$= \sum_i \sum_j f(x_i, y_j)\, P(A \cap \{X = x_i\} \cap \{Y = y_j\})$$

$$= \sum_i \sum_j f(x_i, y_j)\, P(X = x_i)\, P(A \cap \{Y = y_j\})$$

Given $A \in \mathcal{F}_n$. $Y$ is $\mathcal{F}_n$ meas, $X$ is ind of $\mathcal{F}_n$. $\Big\} \Rightarrow \{Y = y_j\} \cap A$

$$= \sum P(X=x_i) \underset{\mathcal{E}_n}{\overset{\uparrow}{\sum}} \underset{\mathcal{F}_n}{\overset{\uparrow}{f(x_i, y_j)}} P(A \cap Y = y_j)$$

$$= \sum_i P(X=x_i) \underbrace{\sum_j \sum_{\omega \in A \cap \{Y = y_j\}}}_{\sum_{\omega \in A}} f(x_i, \cancel{y_j}\, Y(\omega))\, p(\omega)$$

(Uniqueness Lemma from class: $\left.\begin{array}{l} X, Y \text{ are } \mathcal{F}_n \text{ meas.} \\ \& \ \forall A \in \mathcal{F}_n, \ \displaystyle\sum_{\omega \in A} X(\omega)\, p(\omega) = \sum_{\omega \in A} Y(\omega)\, p(\omega) \end{array}\right\} \Rightarrow \cancel{X = Y}$ $P(X=Y) = 1$

HW3 Q6: $\left.\begin{array}{l} X \text{ any RV} \\ Y \to \mathcal{F}_n \text{ meas.} \end{array}\right\} \Rightarrow \underbrace{E(X-Y)^2}_{\text{measures ``distance'' between } X \,\&\, Y.} \geq \underbrace{E(X - E_n X)^2}_{\text{dist between } X \,\&\, E_n X.}$

$$① \quad E(X-Y)^2 = \underbrace{E(X - E_n X + E_n X - Y)^2}$$

$$= \underbrace{E(X-E_nX)^2}_{RHS} + \underbrace{E(E_nX-Y)^2}_{\geq 0} + \underbrace{2E\left[(X-E_nX)(E_nX-Y)\right]}_{\text{Claim } = 0 \,!}$$

Compute $E\left((X-E_nX)\underbrace{(E_nX-Y)}_{\mathcal{F}_n \text{ meas.}}\right) = E\, E_n\left((X-E_nX)\underbrace{(E_nX-Y)}_{\mathcal{F}_n \text{ meas.}}\right)$

$$= E\left[(E_nX-Y)\, E_n(X-E_nX)\right]$$

$$= E\left[(E_nX-Y)\underbrace{\left[E_nX - \cancel{E_n} E_n X\right]}_{0}\right] = 0$$

QED

Trick: For any R.V.s $E E_n Z = EZ$.

Pf ① & Tower prop with $m=0$

Alternatively Pf ②: By def, $\forall A \in \mathcal{F}_n$, $\sum_{\omega \in A} E_n Z(\omega)\, p(\omega) = \sum_{\omega \in A} Z(\omega)\, p(\omega)$

Put $A = \Omega$: $\Rightarrow$ $E(E_n Z) = E Z$. QED

Q1 b) $\varphi$ convex: Want $\varphi(E_n X) \leq E_n \varphi(X)$

Recall ~~HW~~ Jensen's proof: $\varphi$ convex $\Rightarrow$ $\varphi(EX) \leq E\varphi(X)$

Pf: $\varphi$ convex $\Rightarrow$ $\forall x, a$, $\varphi(x) \geq \varphi(a) + (x-a)\varphi'(a)$

Chose $\left.\begin{array}{l} a = E_n X \\ x = X \end{array}\right\} \Rightarrow \varphi(X) = \underbrace{\varphi(E_n X)}_{\text{LHS}} + (X - E_n X)\,\varphi'(E_n X).$

Take $E_n$: $\Rightarrow \underbrace{E_n \varphi(X)}_{\text{RHS}} = \underbrace{\varphi(E_n X)}_{\text{LHS}} + \varphi'(E_n X)\,\underbrace{E_n(X - E_n X)}_{0}$

$E_n \varphi(E_n X) = \varphi(E_n X)$

true because $E_n X$ is $\mathcal{F}_n$ meas.

$E_n\left[(X - E_n X)\,\varphi'(E_n X)\right]$

$= \varphi'(E_n X)\, E_n(X - E_n X)$

$= \varphi'(E_n X)\; 0.$

Q4 $X$ $Y$ 2 RV's. $EX = EY = 0$

$X$ indep of $\mathcal{G}_n$
$Y$ $\mathcal{F}_n$ meas
$Z = XY$

$\Rightarrow E_n Z = E_n XY = 0$

(Pf: $E_n Z = Y E_n X = 0$)