Lecture 8 (9/20). Please Enable Your Video If you Can

Last time: Stochastic process ~~is~~ X_n is a RV $\forall n$.

$\boxed{\text{Adapted}}$: $\forall n$, need X_n to be $\mathcal{F}_n$ meas.

(i.e. X_1 is $\mathcal{F}_1$ meas
X_2 " $\mathcal{F}_2$ "
etc.) All trading strats & Stock prices etc.

Intuition : Martingale $\longrightarrow$ "Fair Game"

$M_n \longrightarrow$ adapted Stochastic process.

$\hookrightarrow$ "Winnings at time n after playing a game"

Q : $\to$ Walk away at time n with $ M_n in hand.

$\to$ Play one more $\leftarrow$ $\boxed{\text{Keep playing if game is "fair"}}$

At time n say first n coins came up

$\omega_1, \omega_2 \cdots \omega_n$.

Let $\omega = (\omega_1, \omega_2 \cdots \omega_n, \underline{* \quad * \quad \cdots})$

Cash in hand for this seq at time n $= M_n(\omega)$

Expected return if I play one more, given the first n coins are $(\omega_1, \cdots \omega_n)$: $\boxed{E_n M_{n+1}}$

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If $\underbrace{E_n M_{n+1}}$ $= M_n$ $\leftarrow$ Game is fair!

Definition 5.41. We say an adapted process M_n is a martingale if $\boldsymbol{E}_n M_{n+1} = M_n$. (Recall $\boldsymbol{E}_n Y = \boldsymbol{E}(Y \mid \mathcal{F}_n)$.)

Remark 5.42. Intuition: A martingale is a "fair game".

Example 5.43 (Unbiased random walk). If $X_1, \dots, X_N$ are i.i.d. and mean zero, then $S_n = \sum_{k=1}^n X_k$ is a martingale.

indep & identically dist.

$E X_i = 0$

Eg: $X_n = \begin{cases} +1 & \text{if } n^{th} \text{ coin flips heads} \\ -1 & \text{'' '' '' '' tails} \end{cases}$ (iid & $P(\text{heads}) = P(\text{tails}) = \frac{1}{2}$)

S_n = cumulative winnings after time n.

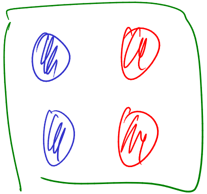
Intuition → seems like a fair game.

Math: $E_n(S_{n+1}) \overset{\text{NTS}}{=\!=} S_n$

$$E_n(S_{n+1}) = E_n(S_n + X_{n+1})$$
$$= E_n S_n + E_n X_{n+1}$$
$$= S_n + \underbrace{E X_{n+1}}_{0 \text{ by assumption.}} \quad (\because X_{n+1} \text{ is ind of } (X_n))$$

$\Rightarrow E_n S_{n+1} = S_n \Rightarrow S$ is a mg.

Example 5.44 (Drawing balls without replacement). Red or Blue balls are drawn from a container *without replacement*. The container has 2 red and 2 balls initially. You win \$1 if the ball is blue, and lose \$1 if the ball is red. Is the process of your winnings a martingale?

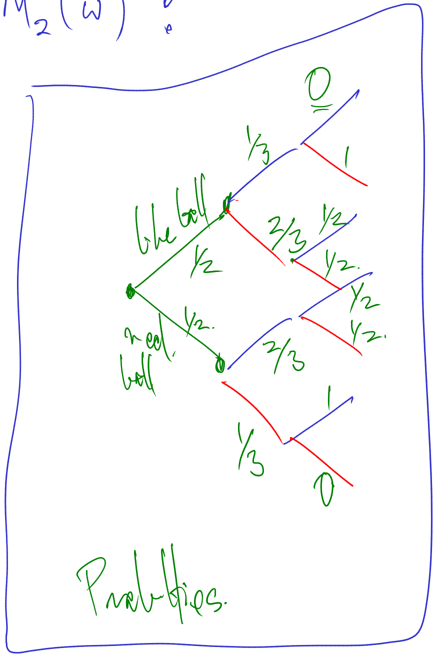


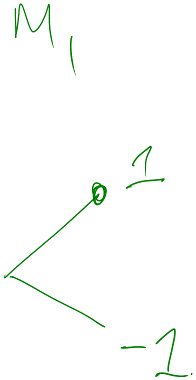
Guess : Not a Mg.

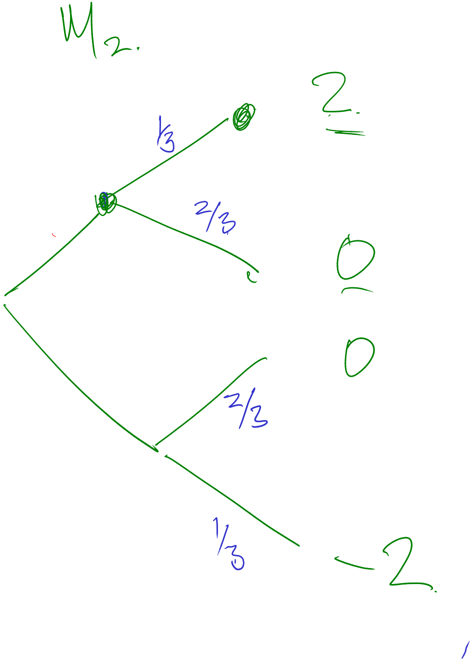
Compute $E_1 M_2$ $\longrightarrow$ $M_n =$ total winnings after time n.

$$M_1(\omega) = \begin{cases} 1 & \omega_1 = \text{blue} \\ -1 & \omega_1 = \text{red} \end{cases}$$

$M_2(\omega)$:


blue ball
red ball
1/2
1/2
1/3
2/3
2/3
1/3
1/2
1/2
1/2
1/2
0
1
1
0
Probabilities.


M_1
1
-1.


M_2.
1/3
2/3
2/3
1/3
2.
0
0
-2.

Compute $E_1 M_2$?

$2\left(\frac{1}{3}\right) + 0 = \frac{2}{3}$

$-\frac{2}{3}$

1

-1

$E_1 M_2 \neq M_1$

What a mug!

Question 5.45. *If M is a martingale, and $m \leqslant n$, is $\boldsymbol{E}_m M_n = M_m$?* $\leftarrow$ Yes (Tower + induction).

Knows $E_n M_{n+1} = M_n$ (def of mg)

$E_m M_{m+1} = M_m.$ ( )

Q: Compute $E_m M_{m+2}$:

def of mg with $n = m+1$

$$E_m M_{m+2} \overset{\text{(tower)}}{=} E_m E_{m+1} M_{m+2} = E_m M_{m+1} \overset{\text{def of mg with } n=m}{=} M_m$$

$$\boxed{\text{By induction } E_m M_n = M_m \ \forall m \leq n.}$$

Question 5.46. *If M is a martingale does $\boldsymbol{E}M_n$ change with n?*

Guess: Does not change with n.

Lemma: For any RV Z, $\forall n$. $\boxed{E Z = E\, E_n Z}$ Lemma.

Pf: Know $\forall A \in \mathcal{F}_n$, $\sum_{\omega \in A} E_n Z(\omega) p(\omega) = \sum_{\omega \in A} Z(\omega) p(\omega)$. (*)

Know $\Omega \in \mathcal{F}_n$ $\forall n$.

Choose $A = \Omega$: $\Rightarrow E\, E_n Z = \sum_{\omega \in \Omega} E_n Z(\omega) p(\omega)$

$$(*) \quad \sum_{\omega \in \Omega} Z(\omega)\, p(\omega) = E Z.$$

QED.

Now Back to Gen Question.

$$\forall n, \quad E M_{n+1} \underset{\substack{\uparrow \\ \text{Lemma}}}{=} E\, E_n M_{n+1} \underset{\substack{\uparrow \\ \text{Def of } M_n}}{=} E M_n.$$

$$\boxed{\Rightarrow \quad E M_{n+1} = E M_n}$$

QED.

Question 5.47. *Conversely, if $\boldsymbol{E}M_n$ is constant, is M a martingale?*

Claim No.

$$\boldsymbol{E}M_2 = \boldsymbol{E}M_1 \overset{\not}{\Rightarrow} \boldsymbol{E}_1 M_2 = M_1$$

$\Leftarrow$ true

M_1

Q: $X_1, X_2, \ldots$ are all ind

Is the proc X a mg? $\boxed{\text{Not a Mg}}$

$$E_n X_{n+1} \underset{\text{indep}}{=} \underbrace{E X_{n+1}}_{\neq X_n} \leftarrow \text{const.}$$

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$$\bigcup_{i,j} \{X = x_i , Y = y_j\} \leftarrow \text{covers all of } \Omega.$$

$$\bigcup_{i,j} \left[\{X = x_i , Y = y_j\} \cap A\right] \leftarrow \text{covers all of } A$$