Lecture 8 (9/17). Please enable your video if you can.

Last time: $X \to$ R.V., $n \leq N$. $E_n X =$ cond exp of X given $\mathcal{F}_n$.

$E_n X$ is the unique RV such that

① $E_n X$ is $\mathcal{F}_n$ meas

& ② $\forall A \in \mathcal{F}_n$, $\displaystyle\sum_{\omega \in A} E_n X(\omega)\, p(\omega) = \sum_{\omega \in A} X(\omega)\, p(\omega)$

Theorem 5.31. *(1) If X, Y are two random variables and $\alpha \in \mathbb{R}$, then $\boldsymbol{E}_n(X + \alpha Y) = \boldsymbol{E}_n X + \alpha \boldsymbol{E}_n Y$. (On homework).*
(2) (Tower property) *If $m \leqslant n$, then $\boldsymbol{E}_m(\boldsymbol{E}_n X) = \boldsymbol{E}_m X$.*
(3) If X is $\mathcal{F}_n$ measurable, and Y is any random variable, then $\boldsymbol{E}_n(XY) = X\boldsymbol{E}_n Y$.

(2) last time

Pf of (3): To show $X E_n Y = E_n(XY)$ we only NTS

(a) $X E_n Y$ is $\mathcal{F}_n$ - meas

(b) $\forall A \in \mathcal{F}_n$, $\sum_{\omega \in A} X(\omega) E_n Y(\omega) \, p(\omega) = \sum_{\omega \in A} X(\omega) Y(\omega) \, p(\omega)$

Pf of (a): X is $\mathcal{F}_n$ - meas by assumption
$E_n Y$ " " " by def
$\}$ product $\Rightarrow X E_n Y$ is $\mathcal{F}_n$ meas

(b) Let $\{x_1, \dots, x_k\} =$ Range of X.

Write $A = \bigcup_{i=1}^{k} \{X = x_i\} \cap A$ (disj union)

$$\sum_{\omega \in A} X(\omega) E_n Y(\omega) \phi(\omega) = \sum_{i=1}^{k} \sum_{\omega \in \underbrace{A \cap \{X = x_i\}}} X(\omega) E_n Y(\omega) \phi(\omega)$$

$$= \sum_{i=1}^{k} \sum_{\omega \in A \cap \{X = x_i\}} x_i \, E_n Y(\omega) \phi(\omega)$$

$$= \sum_{i=1}^{k} x_i \sum_{\omega \in A \cap \{X = x_i\}} E_n Y(\omega) \phi(\omega).$$

$$= \sum_{i=1}^{k} x_i \sum_{\omega \in A \cap \{X = x_i\}} Y(\omega) P(\omega)$$

(∵ $A \cap \{X = x_i\}$ is $\mathcal{F}_n$ meas)

$$= \underbrace{\sum_{i=1}^{k} \sum_{\omega \in A \cap \{X = x_i\}}} x_i \, Y(\omega) P(\omega)$$

$x_i \to X(\omega)$

(∵ X is $\mathcal{F}_n$-meas).

$$= \sum_{\omega \in A} X(\omega) Y(\omega) P(\omega)$$

QED.

Theorem 5.32. *If X is independent of $\mathcal{F}_n$ then $\boldsymbol{E}_n X = \boldsymbol{E} X$.*

(i.e. for every $A \in \mathcal{F}_n$ & $B \subseteq \mathbb{R}$, the events A & B are ind)

($\Leftarrow$ X is ind of $\mathcal{F}_n$, if $\text{Range}(X) = \{x_1, \dots x_k\}$

& $\forall i \in \{1, \dots k\}$ $\forall A \in \mathcal{F}_n$, A & $\{X = x_i\}$ are ind)

Pf: Assume X ind of $\mathcal{F}_n$.

NTS $E_n X = EX$.

i.e. NTS ① EX is $\mathcal{F}_n$ meas -

(b) $\forall A \in \mathcal{F}_n$, $\underbrace{\sum_{\omega \in A} EX\, p(\omega)} = \underbrace{\sum_{\omega \in A} X(\omega)\, p(\omega)}$.

Pf of (a): Note EX is a const R.V. (ie $\forall \omega \in \Omega$, the R.V. takes on the value $EX \in \mathbb{R}$).

EX is $\mathcal{F}_n$ meas $\forall n$. ✓

Pf of (b): Start with RHS. Let $\text{Range}(X) = \{x_1, \dots x_k\}$

$$A = \bigcup_{i=1}^{k} A \cap \{X = x_i\}.$$

$$\sum_{\omega\in A} X(\omega)\,p(\omega) = \sum_{i=1}^{k} \sum_{\omega\in A\cap\{X=x_i\}} X(\omega)\,p(\omega)$$

x_i

$$= \sum_{i=1}^{k} x_i \sum_{\omega\in A\cap\{X=x_i\}} p(\omega)$$

$$= \sum_{i=1}^{k} x_i \boxed{P(A\cap\{X=x_i\})}$$

$$= \sum_{i=1}^{k} x_i\, P(A)\, P(X=x_i)$$

($\because$ X is ind of $\mathcal{E}_n$ $\Rightarrow$ A & $\{X=x_i\}$ are ind.)

$$= P(A) \sum_{i=1}^{k} x_i P(X = x_i)$$

$$= P(A) \cdot EX = \sum_{\omega \in A} EX \, P(\omega) . \quad \boxed{QED.}$$

Intuition so far: Computing E_n

- δ_n mess things → leave alone
- δ_n → indep things → average

Theorem 5.33 (Independence lemma)**.** *If X is independent of $\mathcal{F}_n$ and Y is $\mathcal{F}_n$-measurable, and $f : \mathbb{R} \to \mathbb{R}$ is a function then*

$$\boldsymbol{E}_n f(X, Y) = \sum_{i=1}^{m} f(x_i, Y) \boldsymbol{P}(X = x_i)\,, \qquad \text{where } \{x_1, \dots, x_m\} = X(\Omega)\,.$$

On HW.

$\text{Range}(X) = \{x_1, \dots x_m\}$, $\boxed{y \in \mathbb{R}}$ (a real #).

$$E f(X, y) = \sum_{i=1}^{m} f(x_i, y) P(X = x_i)$$

5.4. **Martingales.**

Definition 5.34. A *stochastic process* is a collection of random variables $X_0, X_1, \ldots, X_N$.

Definition 5.35. A stochastic process is *adapted* if X_n is $\mathcal{F}_n$-measurable for all n. (Non-anticipating.)

Question 5.36. *Is* $X_n(\omega) = \sum_{i \leqslant n} \omega_i$ *adapted?*

Question 5.37. *Is* $X_n(\omega) = \omega_n$ *adapted? Is* $X_n(\omega) = 15$ *adapted? Is* $X_n(\omega) = \omega_{15}$ *adapted? Is* $X_n(\omega) = \omega_{N-i}$ *adapted?*

Remark 5.38. We will always model the price of assets by *adapted* processes. We will also only consider trading strategies which are adapted.

$$X_n(\omega) = \sum_{i=1}^{n} \omega_i \qquad (\omega_i \in \{\pm 1\}.)$$

1 → Stochastic process? Yes

2 X_n adapted?

$X_1(\omega) = \omega_1 \leftarrow \mathcal{F}_1$ meas

$X_2(\omega) = \omega_1 + \omega_2 \leftarrow \mathcal{F}_2$ meas

$X_3(\omega) = \omega_1 + \omega_2 + \omega_3 \leftarrow \mathcal{F}_3$ meas

} works thm. ✓

Example 5.39 (Money market). Let $Y_0 = Y_0(\omega) = a \in \mathbb{R}$. Define $Y_{n+1} = (1+r)Y_n$. (Here r is the interest rate.)

Example 5.40. Suppose $\Omega = \{\pm 1\}^N \cong \{H,T\}^N \cong \{1,2\}^N$. Let $S_0 = a \in \mathbb{R}$. Define $S_{n+1}(\omega) = \begin{cases} uS_n(\omega) & \omega_{n+1} = 1\,, \\ dS_n(\omega) & \omega_{n+1} = -1\,. \end{cases}$

Is S_n adapted? (Used to model stock price in the multi-period Binomial model.)

Definition 5.41. We say an adapted process M_n is a martingale if $\boxed{\boldsymbol{E}_n M_{n+1} = M_n.}$ (Recall $\boldsymbol{E}_n Y = \boldsymbol{E}(Y \mid \mathcal{F}_n)$.)

Remark 5.42. Intuition: A martingale is a "fair game".

Example 5.43 (Unbiased random walk). If $X_1, \dots, X_N$ are i.i.d. and mean zero, then $S_n = \sum_{k=1}^n X_k$ is a martingale.