Lecture 7 (9/15). Please enable your video if you can.

Last time: $E_n X(\omega)$ = cond. exp of X given $\mathcal{F}_n$

$$= \frac{1}{P(\Pi_n(\omega))} \sum_{\omega' \in \Pi_n(\omega)} X(\omega') p(\omega')$$

= Average of X on the event $\Pi_n(\underline{\omega})$

= Avg of X ~~on~~ over all possible future coin tosses given that the first n have come up $\omega_1, \omega_2 \cdots \omega_n$.

Proposition 5.27 (Uniqueness). *If Y and Z are two $\mathcal{F}_n$-measurable random variables such that $\sum_{\omega \in A} Y(\omega)p(\omega) = \sum_{\omega \in A} Z(\omega)p(\omega)$ for every $A \in \mathcal{F}_n$, then we must have $\boldsymbol{P}(Y = Z) = 1$.*

Note avg of Y on $A = \frac{1}{P(A)} \sum_{\omega \in A} Y(\omega)p(\omega)$

" " Z on $A = \frac{1}{P(A)} \sum_{\omega \in A} Z(\omega)p(\omega)$

equal $\Leftarrow$

Pf: Let $A = \{Y > Z\} = \{\omega \in \Omega \mid Y(\omega) > Z(\omega)\}$.

Q1: $A \in \mathcal{F}_n$? (Yes because Y & Z are both $\mathcal{F}_n$ meas).

$\Rightarrow$ By assumption $\sum_{\omega \in A} Y(\omega)p(\omega) = \sum_{\omega \in A} Z(\omega)p(\omega)$

m

Note $\forall \omega \in A$ $\overline{Y(\omega) > Z(\omega)}$

Only possible if $f(\omega) = 0 \ \forall \omega \in A$ (or $\underline{A = \phi}$)

$\Rightarrow P(A) = 0$

$$\left.\begin{array}{l} \Rightarrow P(Y > Z) = 0 \\ \text{Similarly can show } P(Y < Z) = 0 \end{array}\right\} \begin{array}{l} \Rightarrow P(Y \neq Z) = 0 \\ \Leftrightarrow P(Y = Z) = 1 \end{array}$$

QED.

Definition 5.28. Let X be a random variable, and $n \leqslant N$. We define the *conditional expectation of X given $\mathcal{F}_n$*, denoted by $\boldsymbol{E}_n X$, or $\boldsymbol{E}(X \mid \mathcal{F}_n)$, to be the unique *random variable* such that:

(1) $\boldsymbol{E}_n X$ is a $\mathcal{F}_n$-measurable random variable.

(2) For every $A \subseteq \mathcal{F}_n$, we have $\sum_{\omega \in A} \boldsymbol{E}_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$.

Remark 5.29. This is the definition that generalizes to the continuous case. All properties we develop on conditional expectations will only use the above definition, and not the explicit formula.

avg of $E_n X$ on A = avg. of X on A

$\forall A \in \mathcal{F}_n$

Note ⓐ the formula for $E_n X$ given above certainly satisfies ① & ② (last time)

ⓑ Uniqueness: Follows from 5.27.

Example 5.30. Let S_n be the stock price in the binomial model after n periods. Compute $E_1 S_3$, $E_2 S_3$.

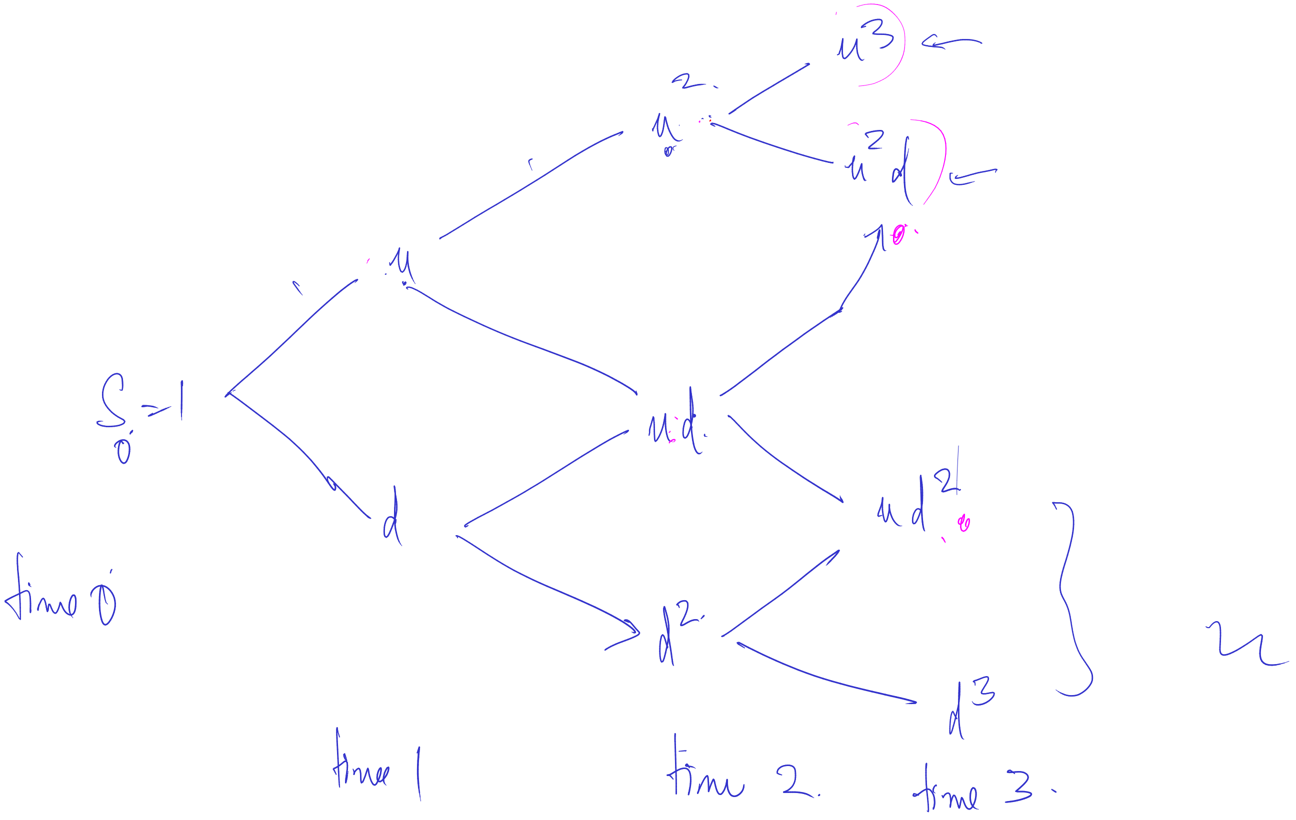
Flip a coin at every time step → Heads with prob p

→ Tails with prob $q = 1-p$.

$$S_{n+1} = \begin{cases} u S_n & \text{if } n+1^{th} \text{ coin flips heads} \\ d S_n & \text{if } (n+1)^{th} \text{ coin flips tails.} \end{cases}$$

$S_0 = 1$.

u n



$S_0 = 1$
u
d
u^2
ud
d^2
u^3
$u^2 d$
ud^2
d^3
time 0
time 1
time 2
time 3.

Find $E_2 S_3$. $\omega = (\omega_1, \omega_2, \omega_3)$ $\omega_i \in \{\pm 1\}$

$\hookrightarrow E_2 S_3$ is an $\mathcal{F}_2 \to$ meas RV.

$$E_2 S_3(\omega) = E_2 S_3(\omega_1, \omega_2, \omega_3)$$ does not dep on ω_3

$\omega = (1, 1, *)$

$\Pi_2(\omega) = \{(1,1,-1), (1,1,+1)\}$.

$P(\Pi_2(\omega)) = p^2 q + p^3 = p^2$

① Find $E_2 S_3(1, 1, *) = u^3 \cdot p + u^2 d \cdot q = \dfrac{p^3 u^3 + p^2 q u^2 d}{p^2}$.

$\to E_2 S_3(1, -1, *) = u^2 d p + u d^2 q$.

$\rightarrow \; E_2 S_3(-1, 1, x) = u^2 d p + u d^2 q$

$\rightarrow \; E_2 S_3(-1, -1, x) = u d^2 p + d^3 \cdot q.$

Claim: $\quad E_2 S_3 = (p u + q d) S_2$

Theorem 5.31. *(1) If X, Y are two random variables and $\alpha \in \mathbb{R}$, then $\boldsymbol{E}_n(X + \alpha Y) = \boldsymbol{E}_n X + \alpha \boldsymbol{E}_n Y$. (On homework)*
(2) (Tower property) *If $m \leqslant n$, then $\boldsymbol{E}_m(\boldsymbol{E}_n X) = \boldsymbol{E}_m X$.*
(3) If X is $\mathcal{F}_n$ measurable, and Y is any random variable, then $\boldsymbol{E}_n(XY) = X\boldsymbol{E}_n Y$.

X & Y 2RV's : $E(XY) \neq X\,\underbrace{EY}_{RV.}$

$E_n(XY) = X E_n Y$

Say X is $\mathcal{F}_n$ - meas.

m

Pf of Tower: NTS $m \le n$. then $E_m(E_n X) = E_m X$.

① Will show ① $E_m(E_n X)$ is $\mathcal{F}_m$ meas. ← True because $E_m(\text{anything})$ is $\mathcal{F}_m$ meas.

& ② $\forall A \in \mathcal{F}_m$, $\sum_{\omega \in A} E_m(E_n X)(\omega)\, p(\omega) = \sum_{\omega \in A} X(\omega)\, p(\omega)$.

Pf: $A \in \mathcal{F}_m$, $\sum_{\omega \in A} E_m(E_n X)(\omega)\, p(\omega) \overset{\text{Def of } E_m}{=} \sum_{\omega \in A} E_n X(\omega)\, p(\omega)$

$\overset{A \in \mathcal{F}_m \subseteq \mathcal{F}_n}{=} \sum_{\omega \in A} X(\omega)\, p(\omega)$ QED.

Def of $E_{\underline{m}}$: $\forall A \in \mathcal{F}_{\underline{m}}$, & any RV Y,

$$\sum_{\omega \in A} \underline{E_{m}} Y(\omega)\, p(\omega) = \sum_{A} \underline{Y(\omega)\, p(\omega)}$$

Theorem 5.32. *If X is independent of $\mathcal{F}_n$ then $\boldsymbol{E}_n X = \boldsymbol{E} X$.*

Theorem 5.33 (Independence lemma)**.** *If X is independent of $\mathcal{F}_n$ and Y is $\mathcal{F}_n$-measurable, and $f\colon \mathbb{R} \to \mathbb{R}$ is a function then*

$$\boldsymbol{E}_n f(X, Y) = \sum_{i=1}^{m} f(x_i, Y)\boldsymbol{P}(X = x_i)\,, \qquad \text{where } \{x_1, \dots, x_m\} = X(\Omega)\,.$$

5.4. **Martingales.**

Definition 5.34. A *stochastic process* is a collection of random variables $X_0, X_1, \ldots, X_N$.

Definition 5.35. A stochastic process is *adapted* if X_n is $\mathcal{F}_n$-measurable for all n. (Non-anticipating.)

Question 5.36. *Is $X_n(\omega) = \sum_{i \leqslant n} \omega_i$ adapted?*

Question 5.37. *Is $X_n(\omega) = \omega_n$ adapted? Is $X_n(\omega) = 15$ adapted? Is $X_n(\omega) = \omega_{15}$ adapted? Is $X_n(\omega) = \omega_{N-i}$ adapted?*

Remark 5.38. We will always model the price of assets by *adapted* processes. We will also only consider trading strategies which are adapted.

Example 5.39 (Money market). Let $Y_0 = Y_0(\omega) = a \in \mathbb{R}$. Define $Y_{n+1} = (1+r)Y_n$. (Here r is the interest rate.)

Example 5.40. Suppose $\Omega = \{\pm 1\}^N \cong \{H, T\}^N \cong \{1, 2\}^N$. Let $S_0 = a \in \mathbb{R}$. Define $S_{n+1}(\omega) = \begin{cases} uS_n(\omega) & \omega_{n+1} = 1\,, \\ dS_n(\omega) & \omega_{n+1} = -1\,. \end{cases}$

Is S_n adapted? (Used to model stock price in the multi-period Binomial model.)

Definition 5.41. We say an adapted process M_n is a martingale if $\boldsymbol{E}_n M_{n+1} = M_n$. (Recall $\boldsymbol{E}_n Y = \boldsymbol{E}(Y \mid \mathcal{F}_n)$.)

Remark 5.42. Intuition: A martingale is a "fair game".

Example 5.43 (Unbiased random walk). If $X_1, \dots, X_N$ are i.i.d. and mean zero, then $S_n = \sum_{k=1}^n X_k$ is a martingale.

Question 5.44. *If M is a martingale, and $m \leqslant n$, is $\boldsymbol{E}_m M_n = M_m$?*

Question 5.45. *If M is a martingale does $\boldsymbol{E}M_n$ change with n?*

Question 5.46. *Conversely, if* $\boldsymbol{E}M_n$ *is constant, is* M *a martingale?*