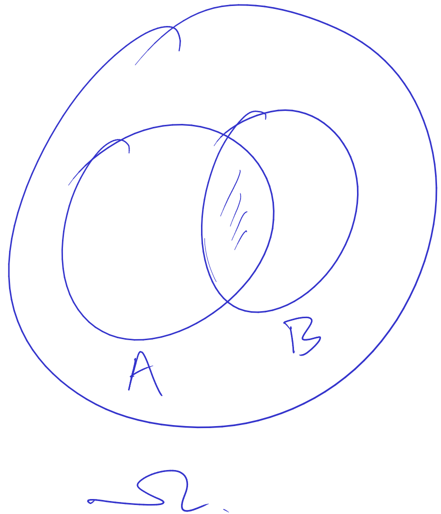
Indep of 2 $\boxed{\text{events}}$

"B occurring does not affect A occurring"



$$\Leftrightarrow \quad \underbrace{P(A|B)}_{\frac{P(A\cap B)}{P(B)} = P(A)} = P(A) \quad \Leftrightarrow \quad P(A\cap B) = P(A)P(B)$$

Events $\longrightarrow$ RV's :

X, Y 2 RVs.

Given X $\longrightarrow$ can observe many events. $\{X > 0\} = \overline{\{\omega \in \Omega \mid X(\omega) > 0\}}$

$\{-1 < X < 1\}$

$\{X \in \mathbb{Z}\}$ etc.

Given Y can observe many events

$\{Y \in \mathbb{Q}\}$, $\{Y \in [1, 2)\}$ etc.

$\boxed{\text{Indep of } X \,\&\, Y} \iff \boxed{\text{every } \text{~~any~~ event you observe using } \underline{X}}$

Lot of work to check

X is indep of every event you observe using Y.

Turns out to be equivalent to checking

$$\longrightarrow \underbrace{P(X = x_i \;\&\; Y = y_j)} = P(X = x_i)\, P(Y = y_j)$$

$\forall\; x_i \in$ Range of X & $y_j \in$ Range of Y

(Only works if Ω is finite, or if X & Y have finite range).

Note $P(X = x_i, Y = y_j) = P(X = x_i) P(Y = y_j)$

$\iff$ The events $\{X = x_i\}$ & $\{Y = y_j\}$ are indep.

Q3. To show equivalence can show (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a)

Your suggestion: (c) $\Rightarrow$ (a)   (a) $\Rightarrow$ (b)

(b) $\Rightarrow$ (a)   (a) $\Rightarrow$ (c)

Lets do (b) $\Rightarrow$ (c) in the case $n = 2$.

Assume X & Y are 2 RV's.

& $\forall A, B \subseteq \mathbb{R}$, $\{X \in A\}$ & $\{Y \in B\}$ are indep.

NTS: Given fns f & g $E f(X) g(Y) = E f(X) \, E g(Y)$.

Say Range of $X = \{x_1, \dots x_m\}$

Range of $Y = \{y_1, \dots y_n\}$

$$E\, f(X)\,g(Y) = \sum_{\omega\in\Omega} P(\omega)\, f(X(\omega))\, g(Y(\omega))$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i)\, g(y_j)\, P(X = x_i, Y = y_j)$$

Choose $A = \{x_i\}$ & $B = \{y_j\}$. Know $\{X\in A\}$ & $\{Y \in B\}$ are indep.

$\Leftrightarrow$ the events $\{X = x_i\}$ & $\{Y = y_j\}$ are indep.

$\Leftrightarrow$ $P(X = x_i, Y = y_j) = P(X = x_i)\, P(Y = y_j)$

Note: ① Ω = disj union $\{X = x_i, Y = y_j\}$

② $$\sum_{\omega \in \Omega} (\ ) = \sum_{i=1}^{m} \sum_{j=1}^{n}$$

$$\sum_{\omega \in \Omega} p(\omega) f(X(\omega)) g(Y(\omega)) = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i) g(y_j) \Big(\underbrace{\sum_{\omega \in \{X = x_i, Y = y_j\}} p(\omega)}_{P(X = x_i, Y = y_j)} \Big).$$

$\hookrightarrow \Rightarrow E\, f(X) g(Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i)\, g(y_j)\, P(X = x_i, Y = y_j)$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i)\, g(y_j)\, P(X = x_i)\, P(Y = y_j)$$

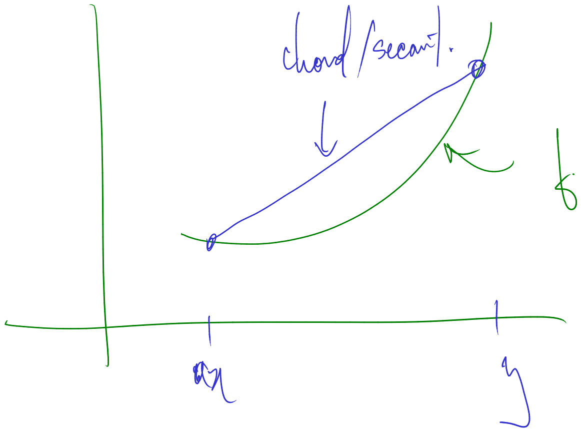
$$= \left(\sum_{i=1}^{m} f(x_i)\, P(X = x_i) \right) \left(\sum_{j=1}^{n} f(y_j)\, P(Y = y_j) \right)$$

$$= E\, f(X) \quad E\, g(Y)$$

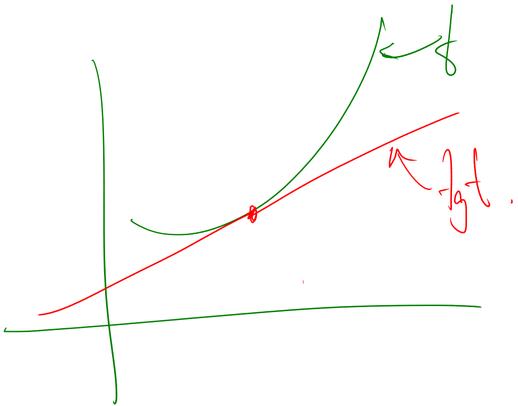
Q4: Convex: f convex

① $f'' \geq 0$ ($\Leftrightarrow$ f' is increasing, if f is twice diff)

②



③ $\Leftrightarrow$ f lies above the tgt.



4a φ convex iff

$$\Leftrightarrow \quad \varphi(x) \geq \varphi(a) + (x-a)\,\varphi'(a) \quad \forall\ x, a \in \mathbb{R}.$$

choose $a = EX$

$$\varphi(X) \geq \varphi(EX) + \varphi'(EX)\left[X - EX\right].$$

take E :
$$E\,\varphi(X) \geq \varphi(EX) + \cancel{\varphi'(EX)\underbrace{\left[EX - EX\right]}_{0}}$$

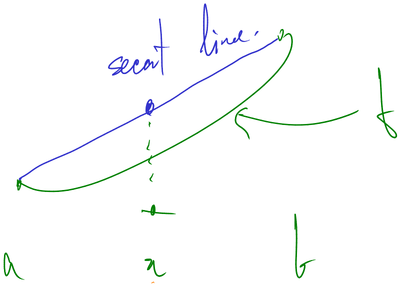
4. Say φ is strictly convex. (Know $E\varphi(X) \geq \varphi(EX)$).

NTS: $E\,\varphi(X) = \varphi(EX) \iff X$ is constant.

Q: What does strictly convex mean?

φ Strictly convex means: $\varphi'' > 0$.

In terms of secants:



$$\forall x \in (a,b), \quad f(x) < \frac{f(b)-f(a)}{(b-a)}(x-a) + f(a)$$

φ convex means $\forall x \in (a,b)$, $f(x) \leq \dfrac{f(b)-f(a)}{b-a}(x-a) + f(a)$,

In terms of tangents:

φ convex means $\forall x, a \in \mathbb{R}$, $f(x) \geq f'(a)(x-a) + f(a)$.

φ strictly convex means $\forall x, a \in \mathbb{R}$ & $x \neq a$, $f(x) > f'(a)(x-a) + f(a)$

For Jensen:

$$\varphi(X) \geq \varphi(EX) + \varphi'(EX)[X - EX].$$

& strict inequality holds when $X \neq EX$.

Q: Say Y & Z are 2 RV's $\ni$ $Y \geq Z$.

① Must $EY \geq EZ$? Yes.

② If $Y > Z$, must $EY > EZ$? Yes

③ If $Y \geq Z$ & $P(Y > Z) > 0$

must $EY > EZ$? Yes.

5b Guess $X+Y$ is $\mathcal{F}_n$ meas. (Yes)

Intuition: $X(\omega)$ only dep on $\omega_1 \cdots \omega_n$

$Y(\omega)$ " " " $\omega_1 \cdots \omega_n$.

$\Rightarrow X(\omega)+Y(\omega)$ only dep on $\omega_1 \cdots \omega_n$

$\Rightarrow X+Y$ is $\mathcal{F}_n$ - meas (Correct intuition, need more for pf)

More for Pf:

Formal Pf: NTS $\forall A \subseteq \mathbb{R}$, $\{X+Y \in A\} \in \mathcal{F}_n$.

$\text{Range}(X) = \{x_1, \dots x_m\}$

$\text{Range}(Y) = \{y_1, \dots y_n\}$.

$$\{X+Y \in A\} = \bigcup_{x_i + y_j \in A} \{X = x_i \ \& \ Y = y_j\}$$

$$= \bigcup_{x_i + y_j \in A} \underbrace{\{X = x_i\}}_{\in \mathcal{F}_n} \cap \underbrace{\{Y = y_j\}}_{\in \mathcal{F}_n}$$

& use ⓐ.

Q3] (a) $X_1 \dots X_n$ indep officially means

$$\boxed{P(X_1 = x_1, \dots X_n = x_n) = P(X_1 = x_1) \cdots P(X_n = x_n) \ \forall x_i \in \mathbb{R}}$$

$\Leftrightarrow$ For $A_1 = \{x_1\}$, $A_2 = \{x_2\}$ ... $A_n = \{x_n\}$ then

mult law holds for the events $\underbrace{\{X_1 \in A_1\}, \dots \{X_n \in A_n\}}$

lets do (3a) $\Rightarrow$ (3b) for $n = 2$.

Given X & Y are indep $\left(\forall x, y \in \mathbb{R} \quad P(X = x, Y = y) = P(X = x) P(Y = y)\right)$

Pick any $A, B \subseteq \mathbb{R}$. NTS $P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B)$

Let $\text{Range}(X) = \{x_1, \dots x_m\}$
$\text{Range}(Y) = \{y_1, \dots y_n\}$.

Then $\{X \in A, Y \in B\} = \underbrace{\bigcup_{\substack{x_i \in A \\ y_j \in B}}}_{\text{disj union}} \{X = x_i, Y = y_j\}$

$$\Rightarrow P(X \in A, Y \in B) = \sum_{\substack{x_i \in A \\ y_j \in B}} P(X = x_i, Y = y_j) = \sum$$

$$= \left(\left(\sum_{x_i \in A} P(X = x_i) \right) \left(\sum_{y_i \in B} P(Y = y_i) \right) \right)$$ done!