Lecture 6 (9/13). Please enable your video if possible.

(1) Finance use of cond exp:

Given any security with payoff V_N at time N.

(For nice markets) AFP at time $n \le N$

is $\boxed{\text{Conditional exp}}$ of V_N (wrt the "Risk Neutral measure")

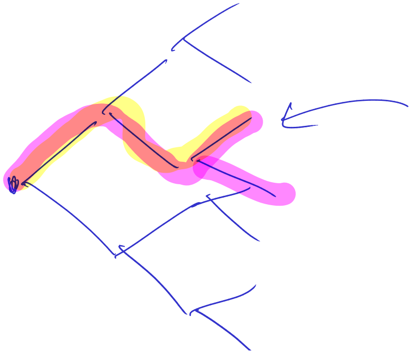
& discounted,

4_1

Recall : $\mathcal{F}_n$ = all events describable by only first n coins

$$= \{ A \subseteq \Omega \mid A = \bigcup_{i=1}^{k} \Pi_n(\omega^i) \text{ for some } k \in \mathbb{N} \ \& \ \omega^1, \dots \omega^k \in \Omega \}.$$

$$\Pi_n(\omega) = \{ \omega' \in \Omega \mid \omega'_i = \omega_i \ \forall i \leq n \}$$



ω = Yellow highlight path.

$\Pi_2(\omega)$ = pink highlighted things.

5.3. Conditional expectation.

Definition 5.23. Let X be a random variable, and $n \leqslant N$. We define $\boldsymbol{E}(X \mid \mathcal{F}_n) = \boldsymbol{E}_n X$ to be the *random variable* given by

$$\boldsymbol{E}_n X(\omega) = \frac{\displaystyle\sum_{\omega' \in \Pi_n(\omega)} p(\omega') X(\omega')}{\displaystyle\sum_{\omega' \in \Pi_n(\omega)} p(\omega')}, \qquad \text{where} \qquad \Pi_n(\omega) = \{\omega' \in \Omega \mid \omega'_1 = \omega_1, \ \dots, \omega'_n = \omega_n\}$$

Remark 5.24. $\boldsymbol{E}_n X$ is the "best approximation" of X given only the first n coin tosses.

Remark 5.25. The above formula does not generalize well to infinite probability spaces. We will develop a definition that does generalize; after we have that definition we will never ever ever use this formula.

$$E_n X(\omega) = \text{Avrage of } X \text{ on the event } \Pi_n(\omega)$$

$$= \frac{1}{P(\Pi_n(\omega))} \underbrace{\sum_{\omega' \in \Pi_n(\omega)} X(\omega')\, p(\omega')}.$$

Proposition 5.26. *The conditional expectation $\boldsymbol{E}_n X$ defined by the above formula satisfies the following two properties:*

(1) $\boldsymbol{E}_n X$ *is an $\mathcal{F}_n$-measurable* random variable.

(2) *For every $A \in \mathcal{F}_n$,* $\displaystyle\sum_{\omega \in A} \boldsymbol{E}_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega).$

Note ② → Average of $E_n X$ on any $\mathcal{F}_n$-meas event A

= Average of X " the same event A.

Note: $A \in \mathcal{F}_n$. Average of $E_n X$ on A $= \dfrac{1}{P(A)} \displaystyle\sum_{\omega' \in A} E_n X(\omega')\, p(\omega')$

" Average of X on A $= \dfrac{1}{P(A)} \displaystyle\sum_{\omega' \in A} X(\omega')\, p(\omega')$

Pf of (1): NTS $\widehat{E_n X}$ is $\mathcal{F}_n$ meas.

i.e. NTS: If $\tilde{\omega} \in \Omega$ is such that $\tilde{\omega}_i = \omega_i \ \forall i \le n$

then $E_n X(\tilde{\omega}) = E_n X(\omega)$

Pf: Note: If $\tilde{\omega}$ is as above

$$\Pi_n(\tilde{\omega}) = \Pi_n(\omega)$$

Hence $E_n X(\tilde{\omega}) = \dfrac{1}{P(\Pi_n(\tilde{\omega}))} \sum_{\omega' \in \Pi_n(\tilde{\omega})} X(\omega') \, p(\omega')$

$$= \frac{1}{P(\Pi_n(\omega))} \sum_{\omega' \in \Pi_n(\omega)} X(\omega')\, p(\omega') = E_n X(\omega)$$

QED.

Proof of (2):

(1) For any $\omega \in \Omega$, $\displaystyle\sum_{\omega' \in \Pi_n(\omega)} \boldsymbol{E}_n X(\omega') p(\omega') = \sum_{\omega' \in \Pi_n(\omega)} X(\omega') p(\omega')$

Proof (2) NTS. $\forall A \in \mathcal{F}_n$, $\displaystyle\sum_{\omega' \in A} E_n X(\omega') p(\omega') = \sum_{\omega' \in A} X(\omega') p_n(\omega')$.

Step (1) above is checking this for $A = \Pi_n(\omega)$.

Note $\forall \omega' \in \Pi_n(\omega)$, $E_n X(\omega') = E_n X(\omega)$ (by part 1).

$\Rightarrow$ LHS $= \displaystyle\sum_{\omega' \in \Pi_n(\omega)} E_n X(\omega') p(\omega')$

$$= \sum_{\omega' \in \Pi_n(\omega)} E_n X(\omega)\, p(\omega') = E_n X(\omega)\, P(\Pi_n(\omega))$$

by formula for $E_n X$

$$\sum_{\omega' \in \Pi_n(\omega)} X_n(\omega')\, p(\omega') = \text{RHS}$$

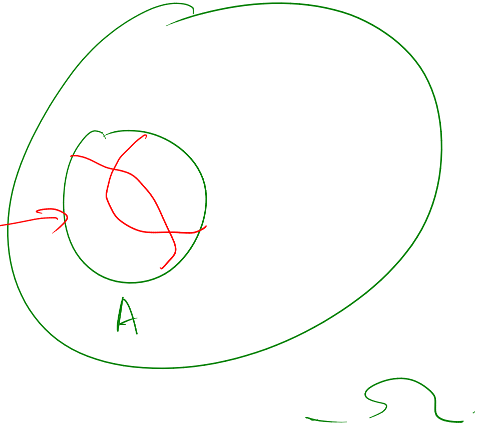
QED.

(2) For any A in Ω, then there exist $\omega^1, \ldots, \omega^k \in \Omega$ such that A is the *disjoint union* of $\Pi_n(\omega^1), \ldots, \Pi_n(\omega^k)$.

$A \in \mathcal{E}_n$

Say (for picture)

$A = \Pi_n(\omega^1) \cup \Pi_n(\omega^2)$



Obs: $\forall\, \omega^1, \omega^2 \in \Omega$.

Either ① $\Pi_n(\omega^1) = \Pi_n(\omega^2)$ OR ② $\Pi_n(\omega^1) \cap \Pi_n(\omega^2) = \phi$

(3) Hence $\sum_{\omega \in A} \boldsymbol{E}_n X(\omega) p(\omega) = \sum_{i=1}^{k} \sum_{\omega \in \Pi_n(\omega^i)} \boldsymbol{E}_n X(\omega) p(\omega) = \sum_{i=1}^{k} \sum_{\omega \in \Pi_n(\omega^i)} X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$.

QED

Know (1) $\forall \omega$, $\sum_{\omega' \in \Pi_n(\omega)} E_n X(\omega') p(\omega') = \sum_{\omega' \in \Pi_n(\omega)} X(\omega') p(\omega')$.

(2) $A \in \mathcal{F}_n$ is of the form $A = \underbrace{\bigcup_{i=1}^{k} \Pi_n(\omega^i)}$ disjoint union.

Proposition 5.27 (Uniqueness). *If Y and Z are two $\mathcal{F}_n$-measurable random variables such that $\sum_{\omega\in A} Y(\omega)p(\omega) = \sum_{\omega\in A} Z(\omega)p(\omega)$ for every $A \in \mathcal{F}_n$, then we must have $\boldsymbol{P}(Y = Z) = 1$.*

IOU Proof (Next time).

Definition 5.28. Let X be a random variable, and $n \leqslant N$. We define the *conditional expectation of X given $\mathcal{F}_n$*, denoted by $\boldsymbol{E}_n X$, or $\boldsymbol{E}(X \mid \mathcal{F}_n)$, to be the unique *random variable* such that:

(1) $\boldsymbol{E}_n X$ is a $\mathcal{F}_n$-measurable random variable.

(2) For every $A \subseteq \mathcal{F}_n$, we have $\sum_{\omega \in A} \boldsymbol{E}_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$.

Remark 5.29. This is the definition that generalizes to the continuous case. All properties we develop on conditional expectations will only use the above definition, and not the explicit formula.

Note: (1) We have checked the the formula for $E_n X$ above satisfies both conditions (1) & (2)

& (2) Uniqueness: by Prop 5.27.