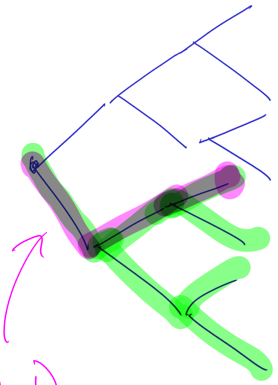
Please enable video if possible.

Office Hours today: End at 5:00 PM.

$\Omega \longrightarrow \{-1, 1\}^N$ (Sample space $\rightarrow$ N coin tosses)

$\omega \in \Omega$. $\omega = (\omega_1, \dots \omega_N) \longrightarrow \omega_i \in \{-1, 1\} \rightarrow$ outcome of ith cointoss

Visualize as trees. $\omega = (-1, 1, 1) \in \Omega$. $(N = 3)$



$(-1, 1, 1)$

$n \in \{0, \dots, N\}$

$\Pi_n(\omega) = \{$ all sample points ω' for which the first n coin tosses agree with those in $\omega \}$.

i.e. $\Pi_n(\omega) = \{\omega' \in \Omega \mid \omega' = (\omega'_1, \dots, \omega'_N)$ & $\omega'_1 = \omega_1,\ \omega'_2 = \omega_2 \dots \omega'_n = \omega_n\}$.

$n = 1$. Draw $\Pi_1(\omega)$

$\omega = (-1, 1, 1)$

5.2. Filtrations.

- Let $N \in \mathbb{N}$, $d_1, \ldots, d_N \in \mathbb{N}$, $\Omega = \{1, \ldots, d_1\} \times \{1, \ldots, d_n\} \times \cdots \times \{1, \ldots, d_N\}$.
- That is $\Omega = \{\omega \mid \omega = (\omega_1, \ldots, \omega_N),\ \omega_i \in \{1, \ldots, d_i\}\}$.
- $d_n = 2$ for all n corresponds to flipping a two sided coin at every time step.

Definition 5.18. We define a *filtration* on Ω as follows:

▷ $\mathcal{F}_0 = \{\emptyset, \Omega\}$.
▷ $\mathcal{F}_1$ = all events that can be described by only the first coin toss (die roll). E.g. $A = \{\omega \mid \omega_1 = H\} \in \mathcal{F}_1$.
▷ $\mathcal{F}_n$ = all events that can be described by only the first n coin tosses.

More precisely, given $\omega = (\omega_1, \ldots, \omega_N) \in \Omega$ and $n \in \{0, \ldots, N\}$ define

$$\Pi_n(\omega) = \{\omega' \in \Omega \mid \omega' = (\omega'_1, \ldots, \omega'_N) \text{ and } \omega'_i = \omega_i \text{ for all } i \leqslant n\}\,.$$

Now $\mathcal{F}_n$ is defined by $\mathcal{F}_n \stackrel{\text{def}}{=} \left\{A \subseteq \Omega \;\middle|\; A = \bigcup_{i=1}^{k} \Pi_n(\omega^i),\ \omega^1, \ldots, \omega^k \in \Omega\right\}$

Remark 5.19. Note $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_N = \mathcal{P}(\Omega)$.

Question 5.20. *Let $\Omega = \{H, T\}^3 \cong \{1, 2\}^3$. What are $\mathcal{F}_0, \ldots, \mathcal{F}_3$?*

↳ $\Omega = \{-1, 1\}^3$

$\left(\begin{array}{l} -1 \longrightarrow \text{tails} \\ +1 \longrightarrow \text{Heads} \end{array}\right)$

$N = 3$

Compute $\mathcal{F}_0 = \{\phi, \Omega\}$

$$\mathcal{F}_1 = \{\phi, \Omega, \{(1,1,1), (1,1,-1), (1,-1,1), (1,-1,-1)\}$$
$$\{(-1,1,1), (-1,1,-1), (-1,-1,1), (-1,-1,-1)\}\}$$

$$\mathcal{F}_2 = \{\phi, \Omega, \{(1,1,1), (1,1,-1)\}, \leftarrow \Pi_2((1,1,1))$$
$$\{(1,1,1), (1,1,-1), (1,-1,1), (1,-1,-1)\}, \leftarrow \Pi_2((1,1,1)) \cup \Pi_2((1,-1,1))$$

& keep going }.

$$\Pi_2(1,1,1) = \{(1,1,-1), (1,1,1)\}$$

$$\Pi_2((1,-1,1)) = \{(1,-1,1), (1,-1,-1)\}$$

Definition 5.21. Let $n \in \{0, \ldots, N\}$. We say a random variable X is $\mathcal{F}_n$-measurable if $X(\omega)$ only depends on $\omega_1, \ldots, \omega_n$.

- Equivalently, for any $B \subseteq \mathbb{R}$, the event $\{X \in B\} \in \mathcal{F}_n$.
- Equivalently, if $\omega' \in \Pi_n(\omega)$ then $X(\omega') = X(\omega)$.

$\{X \in B\} = \{\omega \in \Omega \mid X(\omega) \in B\}$

Question 5.22. *Let $X(\omega) \stackrel{\text{def}}{=} \omega_1 - 10\omega_2$. For what n is $\mathcal{F}_n$-measurable?*

$X(\omega) = \omega_1 - 10\omega_2$ is $\mathcal{F}_n$ meas $\forall n \geq 2$

X is $\mathcal{F}_0$ meas $\iff$ X is constant.

In finance: Always require trading strategies to be "adapted"

i.e. Δ_n = # share of stock in your Pf at time n.

Always need Δ_n to be $\mathcal{F}_n$ measurable.

Cond Prob $\Rightarrow$ $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

Diff from $\mathcal{E}$

5.3. Conditional expectation.

Definition 5.23. Let X be a random variable, and $n \leqslant N$. We define $\boldsymbol{E}(X \mid \mathcal{F}_n) = \boldsymbol{E}_n X$ to be the *random variable* given by

$$(E_n X)(\omega) = \boldsymbol{E}_n X(\omega) = \frac{\displaystyle\sum_{\omega' \in \Pi_n(\omega)} p(\omega') X(\omega')}{\displaystyle\sum_{\omega' \in \Pi_n(\omega)} p(\omega')}, \quad \text{where} \quad \boxed{\Pi_n(\omega)} = \{\omega' \in \Omega \mid \omega'_1 = \omega_1, \ \ldots, \omega'_n = \omega_n\}$$

Remark 5.24. $\boldsymbol{E}_n X$ is the "best approximation" of X given only the first n coin tosses.

Remark 5.25. The above formula does not generalize well to infinite probability spaces. We will develop a definition that does generalize; after we have that definition we will never ever ever use this formula.

$E_n X = E(X \mid \mathcal{F}_n) =$ condition expectation of X Given $\mathcal{F}_n$.

given

NOTE: $E_n X$ is a $\mathcal{F}_n$-meas Random Variable

Note $E_n X(\omega) =$ avg of X over the event $\Pi_n(\omega)$ $= \dfrac{1}{P(\Pi_n(\omega))} \displaystyle\sum_{\omega' \in \Pi_n(\omega)} p(\omega') X(\omega')$

Proposition 5.26. *The conditional expectation $\boldsymbol{E}_n X$ defined by the above formula satisfies the following two properties:*

(1) $\boldsymbol{E}_n X$ *is an* $\mathcal{F}_n$*-measurable* random variable.

(2) *For every* $A \in \mathcal{F}_n$, $\sum_{\omega \in A} \boldsymbol{E}_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$.

Choose $A = \Omega$. Then $EX = \sum_{\omega \in A} X(\omega) p(\omega)$.

By (2) also have $EX = \sum_{\omega \in A} E_n X(\omega) p(\omega) = E(E_n X)$

$\Rightarrow E(E_n X) = EX$