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5. A quick introduction to probability

This is just a quick reminder, and specific to our situation (coin toss spaces). You should have already taken a probability course, or be co-enrolled in one. The only thing we will cover in any detail is conditional expectation.

Let $N \in \mathbb{N}$ be large (typically the maturity time of financial securities).

Definition 5.1. The *sample space* is the set $\Omega = \{(\omega_1, \ldots, \omega_N) \mid \text{each } \omega_i \text{ represents the outcome of a coin toss (or die roll).}\}$

- E.g. $\omega_i \in \{H, T\}$, or $\omega_i \in \{\pm 1\}$.
- Coins / dice don't have to be identical: Pick $M_1, M_2, \ldots, \in \mathbb{N}$, and can require $\omega_i \in \{1, \ldots, M_i\}$.
- Usually in probability the *sample space* is simply a set; however, for our purposes it is more convenient to consider "coin toss spaces" as we defined above.

Definition 5.2. A *sample point* is a point $\omega = (\omega_1, \ldots, \omega_N) \in \Omega \in \Omega$.

Definition 5.3. A *probability mass function* is a function $p \colon \Omega \to [0, 1]$ such that $\sum_{\omega \in \Omega} p(\omega) = 1$.

Definition 5.4. An event is a subset of Ω. Define $\boldsymbol{P}(A) = \sum_{\omega \in A} p(\omega)$.

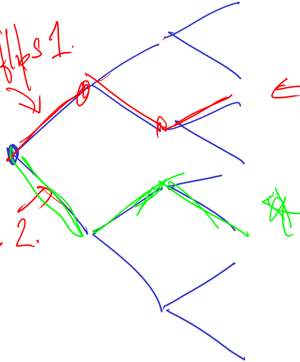
Visualize Ω for coin tosses: $\{\pm 1\}$ or $\{H, T\}$.

N iid coins $\omega_i \in \{1, 2\}$ $\forall i \in \{1, \dots N\}$

independent, identically distributed.

Say $N = 3$. $\omega \in \Omega$. $\omega = (1, 2, 1)$

$\omega = (2, 1, 2)$



5.1. Random Variables and Independence.

Definition 5.5. A *random variable* is a function $X : \Omega \to \mathbb{R}$.

Question 5.6. *What is the random variable corresponding to the outcome of the n^{th} coin toss?*

$$\Omega = \{ \omega = (\omega_1, \ldots \omega_N) \mid \omega_i \in \{1, 2\} \}$$

2 sided coins.

$X_n(\omega)$ = RV corresponding to the n^{th} coin toss.

$= \omega_n$ (where $\omega = (\omega_1, \omega_2, \ldots \omega_N)$)

Definition 5.7. The *expectation* of a random variable X is $\boldsymbol{E}X = \sum X(\omega)p(\omega)$.

Remark 5.8. Note if $\mathrm{Range}(X) = \{x_1, \ldots, x_n\}$, then $\boldsymbol{E}X = \sum X(\omega)p(\omega) = \boxed{\sum_1^n x_i \boldsymbol{P}(X = x_i)}$.

Definition 5.9. The *variance* of a random variable is $\boxed{\mathrm{Var}(X) = \boldsymbol{E}(X - \boldsymbol{E}X)^2}$.

Remark 5.10. Note $\mathrm{Var}(X) = \boxed{\boldsymbol{E}X^2 - (\boldsymbol{E}X)^2}$.

$$\{X = x_i\} = \{\omega \in \Omega \mid X(\omega) = x_i\}$$

Notation convention EX^2 ALWAYS mean $E(X^2)$

and NOT $(EX)^2$.

Definition 5.11. Two events are independent if $\boxed{\boldsymbol{P}(A \cap B) = \boldsymbol{P}(A)\boldsymbol{P}(B).}$ $\Leftarrow$ $P(A \mid B) = P(A)$

Definition 5.12. The events $A_1, \ldots, A_n$ are independent if for any sub-collection $A_{i_1}, \ldots, A_{i_k}$ we have

$$\boldsymbol{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \boldsymbol{P}(A_{i_1})\boldsymbol{P}(A_{i_2}) \cdots \boldsymbol{P}(A_{i_k}).$$

Remark 5.13. When $n > 2$, it is not enough to only require $\boldsymbol{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \boldsymbol{P}(A_1)\boldsymbol{P}(A_2) \cdots \boldsymbol{P}(A_n)$

$\to$ Eg: A_1, A_2, A_3 are ind if ① $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$

& $P(A_1 \cap A_2) = P(A_1)P(A_2)$ & $P(A_1 \cap A_3) = P(A_1)P(A_3)$

& $P(A_2 \cap A_3) = P(A_2)P(A_3)$

Definition 5.14. Two random variables are independent if $\boldsymbol{P}(X = x,\ Y = y) = \boldsymbol{P}(X = x)\boldsymbol{P}(Y = y)$ for all $x, y \in \mathbb{R}$.

Definition 5.15. The random variables $X_1, \dots, X_n$ are independent if for all $x_1, \dots, x_n \in \mathbb{R}$ we have

$$\boldsymbol{P}(X_1 = x_1,\ X_2 = x_2, \dots, X_n = x_n) = \boldsymbol{P}(X_1 = x_1)\boldsymbol{P}(X_2 = x_2) \cdots \boldsymbol{P}(X_n = x_n)\,.$$

Remark 5.16. Independent random variables are uncorrelated, but not vice versa.

Proposition 5.17. *The coin tosses in our setup are all independent, if and only if, there exists functions $p_1, \dots, p_N$ such that*

$$p(\omega) = p_1(\omega_1)p_2(\omega_2) \cdots p_N(\omega_N)\,.$$

Notation convention: Capital letters → RV's

small letters → values they take on.

Pf: Set $X_n(\omega) = \omega_n$ (outcome of nth coin toss).

Coin tosses are indep

$\Leftrightarrow$ The R.V's $X_1, \dots X_n$ are indep.

$\Leftrightarrow$ $\forall\ \omega_1, \omega_2, \dots \omega_N \in \mathbb{R}$ we have

$$\underbrace{P(X_1 = \omega_1 \ \&\ X_2 = \omega_2 \ \dots\ X_N = \omega_N)} = P(X_1 = \omega_1)\, P(X_2 = \omega_2) \cdots P(X_N = \omega_N)$$

$$\| $$

$$P(\{\underbrace{(\omega_1, \omega_2, \dots \omega_N)}_{\omega}\})$$

$$\| $$

$$p(\omega)$$

call this $p_1(\omega_1)$ call this $p_N(\omega_N)$

$$p(\omega) = \underbrace{P(X_1 = \omega_1)}_{p_1(\omega_1)} \cdots \underbrace{P(X_N = \omega_N)}_{p_N(\omega_N)}$$

QED

5.2. Filtrations.

- Let $N \in \mathbb{N}$, $d_1, \dots, d_N \in \mathbb{N}$, $\Omega = \{1, \dots, d_1\} \times \{1, \dots, d_2\} \times \cdots \times \{1, \dots, d_N\}$.
- That is $\Omega = \{\omega \mid \omega = (\omega_1, \dots, \omega_N),\ \omega_i \in \{1, \dots, d_i\}\}$.
- $d_n = 2$ for all n corresponds to flipping a two sided coin at every time step.

$\mathcal{F}_1 \subseteq \mathcal{P}(\Omega)$

power set of Ω.

Definition 5.18. We define a *filtration* on Ω as follows:

▷ $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Info you have before any coin flips.

▷ $\mathcal{F}_1$ = all events that can be described by only the first coin toss (die roll). E.g. $A = \{\omega \mid \omega_1 = H\} \in \mathcal{F}_1$.

▷ $\mathcal{F}_n$ = all events that can be described by only the first n coin tosses.

More precisely, given $\omega = (\omega_1, \dots, \omega_N) \in \Omega$ and $n \in \{0, \dots, N\}$ define

$$\Pi_n(\omega) = \{\omega' \in \Omega \mid \omega' = (\omega'_1, \dots, \omega'_N) \text{ and } \omega'_i = \omega_i \text{ for all } i \leqslant n\}.$$

Now $\mathcal{F}_n$ is defined by $\mathcal{F}_n \stackrel{\text{def}}{=} \left\{ A \subseteq \Omega \;\middle|\; A = \bigcup_{i=1}^{k} \Pi_n(\omega^i),\ \omega^1, \dots, \omega^k \in \Omega \right\}$

Remark 5.19. Note $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_N = \mathcal{P}(\Omega)$.

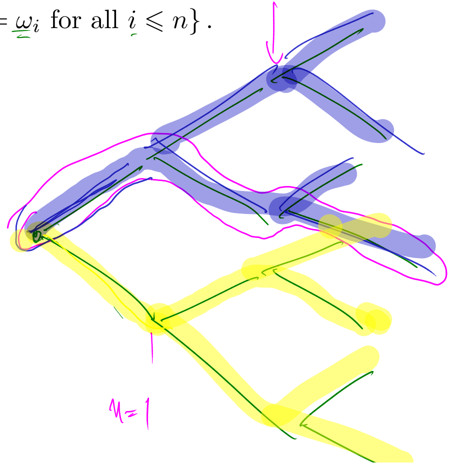
Question 5.20. *Let $\Omega = \{H, T\}^3 \cong \{1, 2\}^3$. What are $\mathcal{F}_0, \dots, \mathcal{F}_3$?*

$\omega^1, \omega^2 \dots \omega^k \to k$ elements of Ω. ω

$\omega^1 = (\omega^1_1, \omega^1_2, \dots \omega^1_N)$

$n = 1$

$\Pi_1(\omega) =$



Definition 5.21. Let $n \in \{0, \dots, N\}$. We say a random variable X is $\boxed{\mathcal{F}_n\text{-measurable}}$ if $\boxed{X(\omega)}$ only depends on $\omega_1, \dots, \boxed{\omega_n}$.

(& not $\omega_{n+1}, \dots \omega_N$)

- ▷ Equivalently, for any $B \subseteq \mathbb{R}$, the event $\{X \in B\} \in \mathcal{F}_n$.
- ▷ Equivalently, if $\omega' \in \Pi_n(\omega)$ then $X(\omega') = X(\omega)$.

Question 5.22. *Let $X(\omega) \stackrel{\text{def}}{=} \omega_1 - 10\omega_2$. For what n is $\mathcal{F}_n$-measurable?*