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## 4. Binomial model (one period)

Say we have access to a money market account with interest rate $r$. The *binomial model* dictates that the stock price varies as follows. Let $p \in (0,1)$, $q = 1 - p$, $0 < d < u$ (up and down factors). Flip a coin that lands heads with probability $p$, and tails with probability $q$. When the coin lands heads, the stock price changes by the factor $u$, and when it lands tails it changes by the factor $d$.

**Question 4.1.** *When is there arbitrage in this market?*

Coin flip

$S_0$ (stock price at time 0)

Heads ($p$) → $u S_0$

Tails ($q = 1-p$) → $d S_0$

Stock

Money market → Interest rate $r$

Market

No Arb $\iff d < 1+r < u$

**Question 4.2.** *If a security pays* $V_1$ *at time* 1*, what is the arbitrage free price at time* 0*. (*$V_1$ *can depend on whether the coin flip is heads or tails).*

Say Binom model  NO arb ( $d < 1+r < u$ ).

Replicate $V_1$. Start with $X_0$ wealth $\begin{cases} \to \Delta_0 \text{ shares of Stock.} \\ \to X_0 - \Delta_0 S_0 \text{ cash.} \end{cases}$

Goal: Find $X_0$ & $\Delta_0$ ∋ wealth at time 1 $=$ $V_1$ (payoff of sec).

If we do this then $X_0$ = AFP.

① $X_1$ = wealth at time 1 $= \Delta_0 S_1 + (X_0 - \Delta_0 S_0)(1+r)$

$$= \Delta_0\left(S_1 - (1+r)S_0\right) + (1+r)X_0 \overset{\text{Want}}{=} V_1.$$

$\Leftrightarrow$ (a) If heads: $\Delta_0\left(uS_0 - (1+r)S_0\right) + (1+r)X_0 = V_1(H)$

(b) If tails: $\Delta_0\left(dS_0 - (1+r)S_0\right) + (1+r)X_0 = V_1(T)$

2 Eq. 2 Unknowns ($X_0$ & $\Delta_0$). Solve

(2) To solve find $\tilde{p}$ & $\tilde{q}$ + $\boxed{\tilde{p} + \tilde{q} = 1}$

$$\& \quad \tilde{p}\,\underbrace{S_1(H)}_{uS_0} + \tilde{q}\,\underbrace{S_1(T)}_{dS_0} = (1+r)S_0$$

$$\boxed{\Leftrightarrow \tilde{p}u + \tilde{q}d = 1+r}$$

$$③ \quad \tilde{p}\,ⓐ + \tilde{q}\,ⓑ \Rightarrow (1+r)X_0 = \tilde{p}V_1(H) + \tilde{q}V_1(T)$$

$$\boxed{\Rightarrow X_0 = \frac{\tilde{p}V_1(H) + \tilde{q}V_1(T)}{1+r}}$$

④ Find $\Delta_0$ : Take ⓐ - ⓑ.

$$\Rightarrow \Delta_0 (u-d) S_0 = V_1(H) - V_1(T)$$

$$\boxed{\Leftrightarrow \Delta_0 = \frac{V_1(H) - V_1(T)}{(u-d) S_0}}$$

⑤ $\tilde{p}u + \tilde{q}d = 1+r \Leftrightarrow \tilde{p}u + (1-\tilde{p})d = 1+r$

$$\Leftrightarrow \tilde{p}(u-d) = 1+r-d \Leftrightarrow \boxed{\tilde{p} = \frac{1+r-d}{u-d}}$$

$$\tilde{q} = 1 - \tilde{p} = \frac{u - (1+r)}{u - d}$$

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Rem: $\tilde{p}$ & $\tilde{q}$ called the "Risk neutral Probabilities".

① Expected return of Stock after time 1.

$$= p\, S_1(H) + q\, S_1(T) = (\underbrace{pu + qd}) S_0$$

② Suppose now the coin flips heads with prob $\tilde{p}$
& tails with prob $\tilde{q}$

Expected return of stock at time $1 = \tilde{p} S_1(H) + \tilde{q} S_1(T)$

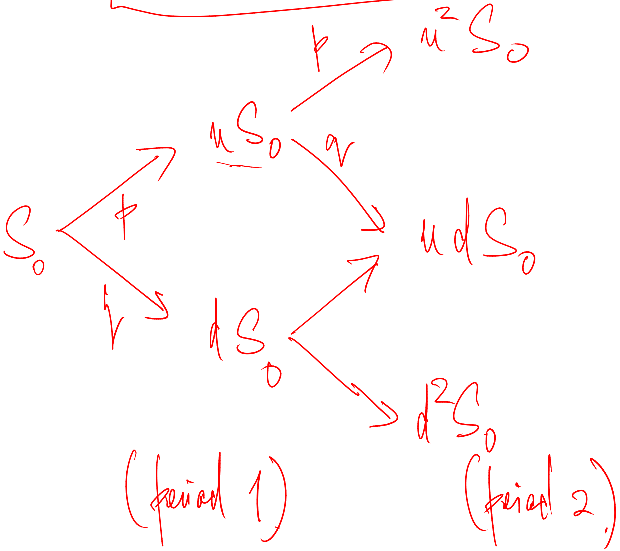
$$= (1+r) S_0$$

= Same return as putting money in bank.

Note AFP $= X_0 = \dfrac{\tilde{p} V_1(H) + \tilde{q} V_1(T)}{\underline{1+r}} = \dfrac{1}{1+r}$ (Expected return of security, if you flip heads with prob $\tilde{p}$ & tails " " $\tilde{q}$).

Expected return under the risk neutral measure.

**Question 4.3.** *What's an N period version of this model? Do we have the same formulae?*




$$S_n = \underbrace{S_0 \left(u^{\#\text{heads}}\right)\left(d^{\#\text{tails}}\right)}_{n \text{ - iid coin flips.}}$$

Goal: Analyze the $n$ period case thoroughly.

1. Securities that don't expire at a fixed time
2. American options.

## 5. A quick introduction to probability

(325 → Coreq. → Either you know prob or you're in 325).

Let $N \in \mathbb{N}$ be large (typically the maturity time of financial securities).

**Definition 5.1.** The *sample space* is the set $\Omega = \{(\omega_1, \dots, \omega_N) \mid \text{each } \omega_i \text{ represents the outcome of a coin toss (or die roll).}\}$

- ▷ E.g. $\omega_i \in \{H, T\}$, or $\omega_i \in \{\pm 1\}$.
- ▷ Coins / dice don't have to be identical: Pick $M_1, M_2, \dots, \in \mathbb{N}$, and can require $\omega_i \in \{1, \dots, M_i\}$.
- ▷ Usually in probability the *sample space* is simply a set; however, for our purposes it is more convenient to consider "coin toss spaces" as we defined above.

**Definition 5.2.** A *sample point* is a point $\omega = (\omega_1, \dots, \omega_N) \in \Omega$.

→ $\omega \in \Omega$ is a sample point & write $\omega = (\omega_1, \omega_2, \dots \omega_N)$

**Definition 5.3.** A *probability mass function* is a function $p \colon \Omega \to [0, 1]$ such that $\sum_{\omega \in \Omega} p(\omega) = 1$.

**Definition 5.4.** An event is a subset of $\Omega$. Define $\boldsymbol{P}(A) = \sum_{\omega \in A} p(\omega)$.

( $p(\omega)$ = prob of $\{\omega\}$ occurring )

$A \subset \Omega$ some event

$P(A)$ = prob $A$ occurs = $\sum_{\omega \in A} p(\omega)$