① HW 1 is online.

② Hint on Q2(f) is on the discussion board

<u>Last time:</u> No arb: To make \$ you need to take risk.

$\Leftrightarrow$ If $X_0 = 0$, know $X_n \geq 0$ then must have $X_n = 0$ (almost surely).

<u>AFP:</u>

$S^{(1)}$
$\vdots$
$S^{(m)}$

Market (arb free)

V ← non traded asset

AFP: Given the opportunity to trade the new asset at price V_0, the market remains arb free!

Question 3.4. *Consider a financial market with a money market account with interest rate r, and a stock. Let $K > 0$. A* forward contract *requires the holder to buy the stock at price K at maturity time N. What is the arbitrage free price at time 0?*

Payoff : Let S_n = stock price at time n.

at maturity forward contract pays $S_N - K$

To compute AFP → Replicate it.

Use only tradable assets, Start with X_0 \$. & end with $(S_N - K)$ \$.

AFP.

Strategy : ① Buy the stock (costs S_0 \$) (worth S_N \$ at time N)

(2) Put $\frac{K}{(1+r)^N}$ \$ in the bank.

(worth K \$ at time N)

(3) Don't trade until maturity.

X_0 = wealth at time 0 = $\left[S_0 - \frac{K}{(1+r)^N} \right]$.

X_N = wealth at time N = $S_N - K$ = payoff of F.C.

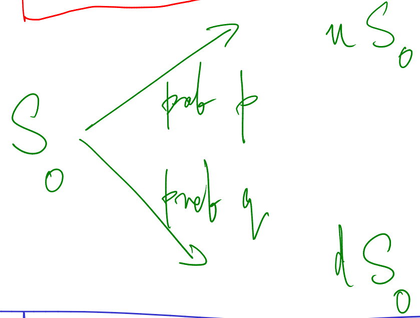
$$\boxed{\Rightarrow \text{AFP of the forward contract at time } 0 \text{ is } S_0 - \frac{K}{(1+r)^N}}$$

4. Binomial model (one period)

Say we have access to a money market account with interest rate r. The *binomial model* dictates that the stock price varies as follows. Let $p \in (0, 1)$, $q = 1 - p$, $0 < d < u$ (up and down factors). Flip a coin that lands heads with probability p, and tails with probability q. When the coin lands heads, the stock price changes by the factor u, and when it lands tails it changes by the factor ~~q~~ d.

Question 4.1. *When is there arbitrage in this market?*

No arb $\iff d < 1+r < u$



If $d \geq 1+r$ then
buy 1 share of stock
Put $-S_0$ \$ in bank.
$\Rightarrow$ Arbitrage

If instead $1+r \geq u$
Short stock & put $-S_0$ \$ in bank
$\Rightarrow$ Arbitrage.

Also need to check there is no arb if $\boxed{d < 1+r < u.}$

Start with $X_0 = 0$ $\begin{cases} \Delta_0 & \text{shares of stock} \\ -\Delta_0 S_0 & \text{in bank.} \end{cases}$

Wealth at time $0 = \Delta_0 S_0 + (-\Delta_0 S_0) = 0$.

Wealth at time $1 = \Delta_0 S_1 - \Delta_0 S_0 (1+r)$

$= \Delta_0 \left(S_1 - (1+r) S_0 \right) = \begin{cases} \Delta_0 \left(u - (1+r)\right) S_0 & \text{if heads} \\ \Delta_0 \left(d - (1+r)\right) S_0 & \text{if tails.} \end{cases}$

Want $X_1 \geq 0$. Note $d < 1+r < u$

$\Rightarrow u-(1+r) > 0$ & $d-(1+r) < 0$.

$X_1 \geq 0$ if heads can only happen if $\Delta_0 \geq 0$

$X_1 \geq 0$ if tails " " " " $\Delta_0 \leq 0$

$X_1 \geq 0$ regardless of coin flip can only happen if $\Delta_0 = 0$.

$\therefore$ If $X_0 = 0$ & $X_1 \geq 0 \Rightarrow X_1 = 0$ (No arb).

$(d < 1+r < u)$

Question 4.2. *If a security pays V_1 at time 1, what is the arbitrage free price at time 0. (V_1 can depend on whether the coin flip is heads or tails).*

Find AFP by replication.

Start with X_0 \$.

→ Δ_0 shares of stock (costs $\Delta_0 S_0$) at time 0.

→ Rest cash. $(X_0 - \Delta_0 S_0)$.

X_1 = wealth at time 1 $= \Delta_0 S_1 + (X_0 - \Delta_0 S_0)(1+r) \overset{\text{Want}}{=} V_1.$

$$X_1 = \Delta_0 \big(S_1 - (1+r) S_0\big) + X_0 (1+r) \overset{\text{Want}}{=} V_1.$$

$S_1(H)$ = stock price at time 1 if heads $= u S_0$

$S_1(T)$ = " " " " 1 " tails $= d S_0$.

$V_1(H)$ = sec price at time 1 if heads

$V_1(T)$ = " " " " " tails.

$$X_1(H) = \Delta_0\big(u-(1+r)\big)S_0 + X_0(1+r) \overset{\text{Want}}{=} V_1(H).$$

$$X_1(T) = \Delta_0\big(d-(1+r)\big)S_0 + X_0(1+r) \overset{\text{Want}}{=} V_1(T).$$

$\boxed{\text{Is this possible?}}$ Can choose Δ_0 & X_0

↳ Yes! (2 eq 2 unknowns) → Δ_0 & X_0

Will solve next time & find Δ_0, X_0.