Math 370 Homework.

Please be aware of the late homework, and academic integrity policies in the syllabus. In particular, you may collaborate, but must write up solutions on your own. You may only turn in solutions you understand.

Assignment 1 (assigned 2021-09-01, due 2021-09-08).

1. Consider a market which only trades a stock and a forward contracts with delivery time 1. The initial price of the stock is $S_0 = \$10$, the forward price $F = \$12$, and at time 1 the stock price S_1 only takes two values: \$11 or \$14. In this market show how you can replicate a call option on the stock with strike price \$13 and maturity time 1. Use this to compute the arbitrage free price of the call.

   Note: A *forward contract* with delivery price K (and delivery time 1) pays $S_1 - K$ at time 1. The *forward price* is that value of K which makes the price of the forward contract \$0 initially.

2. Let $0 < f_1 < f_2 < f_3$, and $r > -1$. Consider a financial market with a stock and a money market account. The money market has interest rate $r > -1$. The stock price changes according to the roll of a fair 3 sided die. If the die rolls $i \in \{1, \dots, 3\}$, then $S_1 = f_i S_0$. (Here S_0 and S_1 are the stock prices at time 1 and time 0 respectively.)
   (a) Find necessary and sufficient conditions on f_1, f_2 and f_3 under which the market has no arbitrage.
   (b) Assuming the market has no arbitrage, find $\tilde{p}_1, \dots \tilde{p}_3 \in (0,1)$ such that
   $$\sum_{i=1}^{3} \tilde{p}_i = 1 \qquad \text{and} \qquad \sum_{i=1}^{3} \tilde{p}_i f_i = (1+r)\,?$$
   (c) Are the numbers $\tilde{p}_i$ in the previous part unique? Prove it, or find more than one such triple of such numbers.
   (d) Suppose now $S_0 = \$1$, $f_1 = 1/2$, $f_2 = 1$ and $f_3 = 2$, and consider a security that pays (at time 1) \$1 if the die rolls 1, \$2 if the die rolls 2, and \$4 if the die rolls 3. Can you replicate this security? If yes, find the (unique) arbitrage free price. If no, is there at least one price at which the security can be traded so that the extended market still has no arbitrage? (In either case prove your answer.)
   (e) Let S_0, f_i be as in the previous part, suppose $r > 0$, and consider a call option on the stock with strike price \$1. (This option would pay \$1 if the die rolls 3, and \$0 otherwise.) Can you replicate this security?
   (f) For the call option in the previous part, find all $V_0 \geqslant 0$ such that introducing this option into the market at price V_0 keeps the market arbitrage free.

Assignment 2 (assigned 2021-09-08, due 2021-09-15).

1. If A, B, C are three events such that $\boldsymbol{P}(A \cap B \cap C) = \boldsymbol{P}(A)\boldsymbol{P}(B)\boldsymbol{P}(C)$, the events A, B are independent, and the events B, C are independent, then must A, B, C be independent? Prove it, or find a counter example.
2. (a) Suppose $X_1, \dots, X_n$ are n independent random variables. True or false: $\operatorname{Var}(\sum_1^n X_i) = \sum_1^n \operatorname{Var}(X_i)$. Prove it, or find a counter example.
   (b) Conversely, if $\operatorname{Var}(\sum_1^n X_i) = \sum_1^n \operatorname{Var}(X_i)$, must $X_1, \dots, X_n$ be independent? Prove it, or find a counter example.
3. Let $X_1, \dots, X_n$ be n random variables. Show that the following are equivalent:
   (a) The random variables $X_1, \dots, X_n$ are independent
   (b) For every collection of sets $A_1 \subseteq \mathbb{R}, \dots, A_n \subseteq \mathbb{R}$, the events $\{X_1 \in A_1\}, \dots, \{X_n \in A_n\}$ are independent.
   (c) For every collection of functions $f_1, \dots, f_n \colon \mathbb{R} \to \mathbb{R}$, we have
   $$\boldsymbol{E}[f_1(X_1) \cdots f_n(X_n)] = \boldsymbol{E} f_1(X_1) \boldsymbol{E} f_2(X_2) \cdots \boldsymbol{E} f_n(X_n)\,.$$
4. *(Jensen's inequality)* Let $\varphi \colon \mathbb{R} \to \mathbb{R}$ be a convex function, and X be a random variable.
   (a) Show that $\varphi(\boldsymbol{E}X) \leqslant \boldsymbol{E}\varphi(X)$. [Hint: Convex functions lie above their tangents.]
   (b) If φ is strictly convex, then show that equality holds in the previous part if and only if X is constant.
   (c) Let $p \geqslant 1$, and X be a random variable. Show that $|\boldsymbol{E}X| \leqslant (\boldsymbol{E}|X|^p)^{1/p}$.
   (d) What happens to the previous part if $p \in (0,1)$?

We will subsequently assume Ω is the probability space corresponding to N die rolls (or coin tosses), and $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_N$ be the filtration corresponding to the die rolls. Given a random variable X, we use the notation $\boldsymbol{E}_n X = \boldsymbol{E}(X \mid \mathcal{F}_n)$ to denote the conditional expectation of X given $\mathcal{F}_n$.

5. (a) Suppose $A, B \in \mathcal{F}_n$. Must $A \cup B \in \mathcal{F}_n$? Prove it. Also, state (without proof) whether $A \cap B$ and A^c must also necessarily belong to $\mathcal{F}_n$?
   (b) Say X and Y are two $\mathcal{F}_n$-measurable random variables. Is $X + Y$ also $\mathcal{F}_n$-measurable? Also, state (without proof) whether $X - Y$ and XY are also $\mathcal{F}_n$-measurable.

Assignment 3 (assigned 2020-09-16, due 2020-09-23).

1. Let X, Y be a random variables.
   (a) For any $\alpha \in \mathbb{R}$ show that $\boldsymbol{E}_n(X + \alpha Y) = \boldsymbol{E}_n X + \alpha \boldsymbol{E}_n Y$ almost surely.
   (b) If $\varphi\colon \mathbb{R} \to \mathbb{R}$ is convex, must $\varphi(\boldsymbol{E}_n X) \leqslant \boldsymbol{E}_n \varphi(X)$ almost surely? Prove it, or find a counter example.
2. Let Ω be a probability space corresponding to N independent coin tosses, where each coin shows heads with probability $p \in (0,1)$ and tails with probability $q = 1 - p$. Let $\omega = (\omega_1, \dots, \omega_N) \in \Omega$, with each $\omega_i \in \{\pm 1\}$ representing the outcome of the i^{th} coin toss. Let $u, d, S_0 > 0$, and define $S_{n+1}(\omega) = u S_n(\omega)$ if $\omega_{n+1} = 1$ and $S_{n+1}(\omega) = d S_n(\omega)$ otherwise.
   (a) Find $\boldsymbol{E}_n S_{n+1}$ and express it in terms of d, u, p, q, n, N and $S_0, \dots, S_n$.
   (b) Let $r > -1$. Find a necessary and sufficient condition on p, u, d, r such that $\boldsymbol{E}_n S_{n+1} = (1 + r) S_n$.
   (c) Given $m, n \in \{0, \dots, N\}$ find a formula for $\boldsymbol{E}_m S_n$. Express your answer in terms of d, u, p, q, m, n. Keep in mind that you are not given $m \leqslant n$.
3. In the previous problem, take $N = 3$. The first two coin tosses are independent and flip heads with probability p and tails with probability q. The third coin toss depends on the first two: If at least one heads is flipped in the first two coin tosses, then the third coin toss flips heads with probability p and tails with probability q. Otherwise the third coin toss flips heads with probability q and tails with probability p. Find $\boldsymbol{E}_2 S_3$ and $\boldsymbol{E}_1 S_3$. (Here p, q, u, d, S are as in the previous part, and you should express your answer in terms of these in a manner to the previous question.)
4. Let X, Y be two random variables such that $\boldsymbol{E}X = \boldsymbol{E}Y = 0$, X is independent of $\mathcal{F}_n$, and Y is $\mathcal{F}_n$-measurable. Let $Z = XY$. Show $\boldsymbol{E}_n Z = 0$. Moreover, show by example, that Z need not be independent of $\mathcal{F}_n$.
5. *(Independence Lemma)* If X is independent of $\mathcal{F}_n$ and Y is $\mathcal{F}_n$-measurable, and $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a function then show

$$\boldsymbol{E}_n f(X, Y) = \sum_{i=1}^{m} f(x_i, Y) \boldsymbol{P}(X = x_i)\,, \qquad \text{where } \{x_1, \dots, x_m\} = X(\Omega)\,.$$

6. *(Least square approximation)* Let X be a random variable, and Y be any $\mathcal{F}_n$-measurable random variable, show that

$$\boldsymbol{E}(X - Y)^2 \geqslant \boldsymbol{E}(X - \boldsymbol{E}_n X)^2\,.$$

Assignment 4 (assigned 2021-09-22, due Never).

In light of your midterm, this homework is optional. These are good problems to use for review and practice, and some of them will be on your next homework.

1. Let Ω be a probability space corresponding to N independent fair coin tosses. Let X_n be the random variable corresponding to the n^{th} coin toss, with $X_n = 1$ corresponding to the coin landing heads and $X_n = -1$ corresponding to the coin landing tails.
   (a) Let $M_n = \sum_{j=1}^{n} X_j$. Is M a martingale? Compute $\boldsymbol{E}M_n$ and $\boldsymbol{E}M_n^2$.
   (b) Let $C, \sigma > 0$ and define $S_n = C^n e^{\sigma M_n}$. Find a relation between C and σ so that S is a martingale.
2. Let X_n be as in the previous problem. Let $a, b \in \mathbb{Z}$, $Y_0 \in (a, b) \cap \mathbb{Z}$ and define $Y_{n+1} = Y_n + X_{n+1}$ if $Y_n \in (a, b)$ and $Y_{n+1} = Y_n$ if $Y_n \in \{a, b\}$.
   (a) Compute $\boldsymbol{E}_n Y_{n+1}$ in terms of $Y_0, \dots, Y_n$.
   (b) Suppose now $a = 0$, $b = 3$, and let $p_n = \boldsymbol{P}(Y_n = 0)$ if $Y_0 = 1$, and let $q_n = \boldsymbol{P}(Y_n = 0)$ if $Y_0 = 2$. Find a relation between p_{n+2} and p_n. Also find a relation between q_{n+2} and q_n.
   (c) Assuming $\lim_{n\to\infty} p_n$ and $\lim_{n\to\infty} q_n$ exist, find $\lim_{n\to\infty} p_n$ and $\lim_{n\to\infty} q_n$.
   (d) *(Optional)* Find p_n and q_n. [We will encounter and solve many such recurrence relations later in the course, and it may be helpful to try your had at it now.]
3. Consider the multi-period binomial model with $u = 2$, $d = 1/2$ and $r = 1/4$. Suppose that the actual probabilities of the coin landing heads and tails are $2/3$ and $1/3$ respectively.
   (a) Find the distribution of S_3 under $\boldsymbol{P}$ and $\tilde{\boldsymbol{P}}$.
   (b) Find the average growth rate of S under $\boldsymbol{P}$ and $\tilde{\boldsymbol{P}}$. (That is find $\boldsymbol{E}S_n/S_0$ and $\tilde{\boldsymbol{E}}S_n/S_0$.
   (c) Suppose $S_0 = \$10$. Consider a call option with strike price \$10 and maturity $n = 2$. Find the arbitrage free price price of this option at times $n = 0$ and $n = 1$.
4. Let Z_N be an $\mathcal{F}_N$-measurable random variable such that $\boldsymbol{E}Z_N = 1$ and $Z_N > 0$. Define a function $\tilde{p}$ by $\tilde{p}(\omega) = Z_N(\omega)p(\omega)$. Since $\boldsymbol{E}Z_N = 1$, we must have $\sum \tilde{p}(\omega) = 1$, and hence $\tilde{p}$ can be viewed as a probability mass function. We will use $\tilde{\boldsymbol{P}}$ to denote the new probability measure obtained from the probability mass function $\tilde{p}$. (Since $Z_N > 0$ note that $\boldsymbol{P}$ and $\tilde{\boldsymbol{P}}$ are equivalent.)
   (a) If X is a random variable show that $\tilde{\boldsymbol{E}}X = \boldsymbol{E}(Z_N X)$.
   (b) Let $Z_n = \boldsymbol{E}_n Z_N$. Show that Z_n is a martingale.
   (c) Must $Z_n > 0$ almost surely? Prove it, or find a counter example.
   (d) Show that $\tilde{\boldsymbol{E}}_n X = \frac{1}{Z_n}\boldsymbol{E}_n(Z_N X)$.
   (e) Consider the N period Binomial model with parameters $0 < d < 1 + r < u$, where each coin flips heads with probability p_1 and tails with probability $q_1 = 1 - p_1$. Let $\tilde{p}_1$ and $\tilde{q}_1$ be the risk neutral probabilities of tossing heads and tails, and $\tilde{\boldsymbol{P}}$ be the risk neutral measure and $\tilde{p}$ the associated probability mass function. Find Z_N so that $\tilde{p}(\omega) = Z(\omega)p(\omega)$, and express $Z_N(\omega)$ in terms of ω, p_1, q_1, $\tilde{p}_1$ and $\tilde{q}_1$. Also find a similar formula for $Z_n = \boldsymbol{E}_n Z_N$.
5. Let $\Omega = \{-1, 1\}^N$, $p_1 \in (0, 1)$, $q_1 = 1 - p_1$, and let $\boldsymbol{P}$ be the probability measure on Ω under which the outcome of each coin toss $\omega_1, \dots, \omega_N$ are i.i.d. with $\boldsymbol{P}(\omega_i = 1) = p_1$. Let u_n, r_n, d_n be adapted processes such that $d_n < 1 + r_n < u_n$, and $r_n > -1$ almost surely. Let $S_0 > 0$, $D_0 = 1$ and define the process D_n and S_n by

$$D_{n+1}(\omega) = (1 + r_n(\omega))^{-1} D_n(\omega)\,, \qquad S_{n+1}(\omega) = \begin{cases} u_n(\omega)S_n(\omega) & \text{if } \omega_{n+1} = 1\,, \\ d_n(\omega)S_n(\omega) & \text{if } \omega_{n+1} = -1\,. \end{cases}$$

   (a) Are D_n and S_n adapted? Are they predictable? Prove it. (A process X is called predictable if for all n, X_{n+1} is $\mathcal{F}_n$ measurable.)
   (b) Is there a measure $\tilde{\boldsymbol{P}}$ under which $D_n S_n$ is a martingale? Prove it.
   (c) Must the coin tosses $\omega_1, \dots, \omega_N$ independent under $\tilde{\boldsymbol{P}}$? Justify.
   (d) Must the coin tosses $\omega_1, \dots, \omega_N$ be identically distributed under $\tilde{\boldsymbol{P}}$? Justify.

   Now consider a market with a stock whose price is modelled by S_n, and a bank with random interest rate r_n (i.e.\$1 cash in the bank at time n becomes $\$(1 + r_n)$ cash in the bank at time $n + 1$).
   (e) Show that the process X is the wealth of a self-financing portfolio if and only if $D_n X_n$ is a martingale under $\tilde{\boldsymbol{P}}$.
   (f) Can there be arbitrage in this market? Prove it.
   (g) Let V_N be an $\mathcal{F}_N$ measurable random variable, and consider a security that pays V_N at maturity. Show that the arbitrage free price of this security at time $n \leqslant N$ is given by $V_n = \tilde{\boldsymbol{E}}_n(D_N V_N / D_n)$.
6. Consider a financial market with a money market account (with interest rate $r = 0$) and a stock with initial price \$100. At time $n + 1$ the stock price either increases by \$10, or decreases by \$10 based on a coin toss. Consider an European call on this stock with strike \$80 and maturity time 5. What is the price of this call at time 0?

Assignment 5 (assigned 2021-09-29, due 2021-10-06).

1. Do questions 1, 3, 4 and 5a, 5b, 5c, 5d from Assignment 4.

Assignment 6 (assigned 2020-10-07, due 2020-10-14).

1. Do Question 5e, 5f, 5g.
2. Do Question 6 from Homework 4.
3. Consider the N-period binomial model with parameters u, d, r such that $0 < d < 1 + r < u$. Let $\hat{\omega} \in \Omega$. The *digital option* (also called the Arrow-Debreu security) pays \$1 if the sequence of coin tosses exactly matches $\hat{\omega}$, and nothing otherwise. Find the arbitrage free price of this security at each time $n \leqslant N$. Also find the number of shares of stock held in the replicating portfolio at each time $n \leqslant N$.
4. Consider the N-period binomial model with $0 < d < 1 + r < u$. Let g be a given function, and consider a security that pays $V_N = g(S_N)$ at maturity time N. Show that the arbitrage free price at time $n \leqslant N$ is

$$\frac{1}{(1+r)^{N-n}} \sum_{k=0}^{N-n} \binom{N-n}{k} \tilde{p}^k \tilde{q}^{N-n-k} f_N(x u^k d^{N-n-k}) .$$

   Note: I stated this in class; and proved it for $n = N - 1$, and $n = N - 2$. Do the general case.
5. Consider the N-period binomial model with $u = 3/2$, $d = 3/4$, $r = 1/4$ and $N = 4$. Suppose $S_0 = \$100$. At maturity $N = 4$ an option pays $(S_4 - S_2)^+$. Find the arbitrage free price at time $n = 0$ and $n = 2$.
6. Consider an N period binomial model with $0 < d < 1 + r < u$. An Asian option has payoff of the form $V_N = g(\sum_0^N S_n)$ for some (non-random) function f. For instance, $g(x) = (\frac{x}{N+1} - K)^+$ corresponds to an Asian call option with strike K. In order to price this security, we introduce a new process $Y_n = \sum_{k=0}^n S_k$. Show that the arbitrage free price of this this security at time n can be expressed in the form $f_n(S_n, Y_n)$, and find an algorithm that computes f_n.

Assignment 7 (assigned 2021-10-13, due 2021-10-20).

1. Consider an N period binomial model with $0 < d < 1 + r < u$. A *variance swap* that expires at time N pays

$$V_N = \frac{1}{N} \sum_{n=1}^{N} \Big(\log\Big(\frac{S_n}{S_{n-1}} \Big) \Big)^2 - K^2 ,$$

   where S_n is the stock price at time n, and K is the strike price. Find a formula for K in terms of the model parameters, such that it costs nothing to enter this contract at time 0.
2. Consider the N period binomial model with $0 < d < 1 + r < u$. We know that there can be no arbitrage at the deterministic time N. Can there be arbitrage at a random time $\tau \leqslant N$? That is, can you find a random time τ and a self financing portfolio for which $X_0 = 0$, $X_\tau \geqslant 0$ and $\boldsymbol{P}(X_\tau > 0) > 0$? Turns out you can't if τ is a stopping time, but can if τ an arbitrary random time.
   (a) Let τ be a finite stopping time and X is the wealth of a self-financing portfolio with $X_0 = 0$ and $X_\tau \geqslant 0$. Let Δ_n be the number of shares of stock held by the portfolio at time n. Define $\Gamma_n = \Delta_n \mathbf{1}_{\{n<\tau\}}$, and consider a new self-financing portfolio Y with initial wealth $Y_0 = 0$ that holds Γ_n shares of stock at time n. Express Y_N in terms of X_τ, and use this to show $X_\tau = 0$. [A simpler way to directly show $X_\tau = 0$ is using the optional sampling theorem, which we will see later.]
   (b) Suppose now that $\tau \colon \Omega \to \{0, \dots, N\}$ is a finite (random) time, and not necessarily a stopping time. Does your proof from the previous part still work? If yes, explain. If no, explain and also find an example where $X_\tau \geqslant 0$, but $\boldsymbol{P}(X_\tau > 0) > 0$.
3. Consider the N period binomial model with $0 < d < 1 + r < u$, and denote the stock price by S_n. Consider an *down and out* option with strike K and barrier price D. If the stock price ever falls below (or equals) D, the option expires worthless. If not, the option allows the holder the right (not obligation) to sell the stock at price K.
   (a) Write down an algorithm/formula to price this option and find the trading strategy required to replicate this option.
   (b) Using your algorithm above and a computer, compute the price of this option at time 0 with $N = 90$, $u = 1.05$, $d = 0.9$, $r = 3\%$, $S_0 = \$10$, $K = \$100$, $D = \$5$. You should also submit the listing of a program that implements this algorithm in your language of choice.
4. Suppose M is an adapted process such that $\boldsymbol{E} M_\sigma = \boldsymbol{E} M_0 = M_0$ for every finite stopping time σ. Must M be a martingale? Prove it, or find a counter example.

Assignment 8 (assigned 2020-10-21, due 2020-10-28).

1. Consider the N period binomial model with $0 < d < 1 + r < u$. The *down and rebate* option with face value F and lower barrier L pays F at the first time the stock price is below (or equal) to L. If the stock price never crosses this barrier, then the option expires worthless.
   (a) Write down an algorithm/formula to price this option and find the trading strategy required to replicate this option.
   (b) Using your algorithm above and a computer, compute the price of this option at time 0 with $N = 100$, $u = 1.1$, $d = 0.9$, $r = 1\%$, $S_0 = \$10$, $F = \$1$, $L = \$8.50$. You should also submit the listing of a program that implements this algorithm in your language of choice. (I have uploaded two Python programs pricing *up and rebate* options. You might find it helpful to look at these and modify them as appropriate / port them to your language of choice.)
   (c) Do the previous part with $N = 5000$. (Note, the code I uploaded is inefficient, and may not finish running for this large N. A better implementation should finish in seconds.)
2. Consider the N period binomial model with $0 < d < 1 + r < u$. A *look-back call option* is a European call option with strike price chosen to be the minimum of the stock price over the entire time period.
   (a) Write down an algorithm/formula to price this option and find the trading strategy required to replicate this option.
   (b) Using your algorithm above and a computer, compute the price of this option at time 0 with $N = 30$, $u = 1.1$, $d = .95$, $r = 2\%$, $S_0 = \$15$. Also compute the number of shares in the replicating portfolio at time 0. You should also submit a snippet of the listing of a program that implements this algorithm in your language of choice.
3. Prove Proposition 6.44 from class, that gives a recurrence relation to price options with a random maturity time.
4. Consider infinitely many i.i.d. coin tosses where the probability of tossing heads is p and probability of tossing tails is $q = 1 - p$. Let $d > 0$ and set $u = 1/d$. We start with $S_0 > 0$ and define $S_{n+1} = uS_n$ if the $(n + 1)^{\text{th}}$ coin is heads, and $S_{n+1} = dS_n$ if the $(n + 1)^{\text{th}}$ coin is tails. Fix $m_0, n_0 \in \mathbb{N}$ and define $L = d^{m_0} S_0$ and $U = u^{n_0} S_0$. Let $\tau = \min\{n \in \mathbb{N} \mid S_n \notin (L, U)\}$.
   (a) Suppose f is a function such that the process $f(S_{\tau \wedge n})$ is a martingale. Find a finite difference equation (FDE) for f.
   (b) If $p \neq q$, find f so that $f(L) = 0$, $f(U) = 1$ and the process $f(S_{\tau\wedge n})$ is a martingale. [HINT: Guess $f(x) = A + Bx^{\alpha}$, and find A, B, α.]
   (c) Do the previous part when $p = q$. [HINT: Try something with $\log x$.]
   (d) Find $\boldsymbol{P}(S_\tau = U)$. [Even though τ is not bounded, one can show that optional sampling theorem still applies. Feel free to assume it.]

Assignment 9 (assigned 2020-10-28, due Never).

1. Consider the N period binomial model with $0 < d < 1 + r < u$, and let S_n denote the stock price at time n. An American put, call and straddle options with strike K have intrinsic value $G_n^P = (K - S_n)^+$, $G_n^C = (S_n - K)^+$ and $G_n^S = |S_n - K|$ respectively. A European put call and straddle with strike K and maturity N have payoffs G_N^P, G_N^C and G_N^S respectively.
   (a) Write down an algorithm/formula to price these options.
   (b) Can you express the arbitrage free price of the European put in terms of the European call and straddle? Justify / explain.
   (c) Are the American options at least as expensive as the European options? If yes, prove it. If no, find a counter example.
   (d) Choose $S_0 = \$8$, $N = 3$, $u = 2$, $d = 1/2$, $r = 1/4$ and $K = 4$. Is there a difference between the price of the American call and European call? Explain.
   (e) With parameters as in the previous part, is there a difference between the price of the American put and European put? Explain.
   (f) With parameters as in the previous part, is the price of the American straddle the sum of that of the American put and American call? Explain.
   (g) Choose $N = 50$, $u = 1.1$, $d = .9$, $r = 1\%$, $S_0 = \$10$, $N = 50$ and $K = (1 + r)^N S_0$. Use a computer to find the prices of the American put and European put at time 0. Also find the number of shares at time 0 held in the replicating portfolio of each of these options. If σ^* is the minimal optimal exercise policy for the American put, find $\max\{S_5(\omega) \mid \omega \in \Omega,\ \sigma^*(\omega) \leqslant 5\}$.
   (h) Do the previous part for $N = 500$.
2. Consider the N period binomial model with $0 < d < 1 + r < u$, and an American option with intrinsic value $G_n = g_n(S_n)$. You are given $g_N(x) = (x - K)^+$, and that *every* exercise policy is optimal. Find finite difference equations for the functions g_n.
3. Consider N i.i.d. coin tosses that land heads with probability p and tails with probability $q = 1 - p$. Let $0 < d < u$, $S_0 > 0$, and define $S_{n+1} = uS_n$ if the $(n + 1)^{\text{th}}$ coin is heads, and $S_{n+1} = dS_n$ otherwise.
   (a) Suppose for some sequence of functions $f_0, f_1, \dots, f_n$, the process $f_n(S_n)$ is a martingale. Find a recurrence relation between these functions.
   (b) Given a function g, use the previous part to write down a system of recurrence relations that allows you to compute $\boldsymbol{E} g(S_N)$.
   (c) Given $\alpha \in \mathbb{R}$ compute $\boldsymbol{E} S_N^\alpha$.
4. You are an election forecaster for a race between two candidates A and B. Let V_n^A be the number votes for A after the first n people have voted, and let $V_n^B = n - V_n^A$, $X_n = V_n^A - V_n^B$. In our model, we dictate that the probability that the $(n + 1)^{\text{th}}$ person has voted for A is V_n^A/n.
   (a) Let $N \in \mathbb{N}$, $n \in \{1, \dots, N\}$, $x \in \{-n, \dots, n\}$ and define $f_n(x) = \boldsymbol{P}(X_N > 0 \mid X_n = x)$. That is, $f_n(x)$ is the chance that candidate A has more votes after N people have voted given that after n people have voted candidate A has x votes more than candidate B. Find f_N, and a recurrence relation that allows you to compute f_n.
   (b) Suppose now $N = 100,000$, and after $10,000$ people have voted candidate A leads by 100 votes. What is the probability he wins, correct to 4 decimal places? (You may use a computer / spreadsheet.)
5. Consider the N period binomial model with $0 < d < 1 + r < u$, and consider an American option with intrinsic value $G = (G_0, G_1, \dots, G_N)$.
   (a) Let X be the wealth of a replicating portfolio (i.e. $X_n \geqslant G_n$ for all n, and $X_{\sigma^*} = G_{\sigma^*}$ for one finite stopping time σ^*). For any finite stopping time σ, let V_0^σ be the arbitrage free price at time 0 of an option with maturity σ and payoff G_σ. Show that $X_0 = \max_\sigma V_0^\sigma$, where the maximum is taken over all finite stopping times σ.
   (b) Let Y be a second replicating portfolio so that $Y_n \geqslant G_n$ for all n and $Y_{\tau^*} = G_{\tau^*}$ for some finite stopping time τ^* (which need not be the same as σ^*). Show that $\mathbf{1}_{n \leqslant (\sigma^* \vee \tau^*)} X_n = \mathbf{1}_{n \leqslant (\sigma^* \vee \tau^*)} Y_n$.
6. Consider the N period binomial model with $0 < d < 1 + r < u$, and consider an American option with intrinsic value $G = (G_0, G_1, \dots, G_N)$. Define $V_N = G_N$, $V_n = \max\{G_n, \tilde{\boldsymbol{E}}_n V_{n+1}/(1 + r)\}$, and $\sigma^* = \min\{n \mid V_n = G_n\}$. Define
$$A_n = \sum_{k=0}^{n-1} \Big(G_k - \frac{1}{1+r} \tilde{\boldsymbol{E}}_k V_{k+1} \Big)^+ (1 + r)^{n-k}, \quad \text{and} \quad X_n = V_n + A_n\,.$$
   (a) Show that $\frac{1}{1+r} A_{n+1} = A_n + (G_n - \frac{1}{1+r} \tilde{\boldsymbol{E}}_n V_{n+1})^+$ and $X_n = \frac{1}{1+r} \tilde{\boldsymbol{E}}_n X_{n+1}$.
   (b) Show that $X_n \geqslant G_n$ for all n and $X_{\sigma^*} = G_{\sigma^*}$.
   (c) Show that $X_n > G_n$ if $n < \sigma^*$.
   (d) Conclude (X_n) is the wealth process of the replicating portfolio of an American option with intrinsic value G, and that σ^* is the minimal optimal exercise time. [We gave an alternate proof of this in class, without using the above explicit formula.]

Assignment 10 (assigned 2021-11-04, due 2021-11-10).

1. Do question 1, 3, 5, 6 from Assignment 9.

Assignment 11 (assigned 2020-11-11, due 2020-11-18).

1. Let $G = (G_0, \dots, G_N)$ be an adapted process, and let V be the Snell super-martingale envelope of G and $V = M - A$ be the Doob decomposition of V. Recall we say that a finite stopping time τ^* solves the optimal stopping problem for G if $\boldsymbol{E}G_{\tau^*} \geqslant \boldsymbol{E}G_\tau$ for all finite stopping times τ.
   (a) If τ^* solves the optimal stopping problem for G, then must $A_{\tau^*} = 0$ and $V_{\tau^*} = G_{\tau^*}$? Prove it, or find a counter example.
   (b) Conversely, if $A_{\tau^*} = 0$ and $V_{\tau^*} = G_{\tau^*}$, must τ^* solve the optimal stopping problem for G? Prove it, or find a counter example.
   (c) Is the previous subpart true if we only assume $V_{\tau^*} = G_{\tau^*}$? Prove it, or find a counter example.
   (d) Is part (b) true if we only assume $A_{\tau^*} = 0$? Prove it, or find a counter example.
   (e) If σ^* and τ^* are two solutions to the optimal stopping problem for G then must $\sigma^* \wedge \tau^*$ also be a solution to the optimal stopping problem for G?
2. Let $G = (G_0, \dots, G_N)$ be an adapted process, and fix $k \in \{0, \dots, N\}$. Let V be the Snell super-martingale envelope of G, and define $\sigma_k^* = \min\{n \geqslant k \mid V_n = G_n\}$. If σ_k is any stopping time such that $\sigma_k \geqslant k$, then show that $\boldsymbol{E}_k G_{\sigma_k^*} \geqslant \boldsymbol{E}_k G_{\sigma_k}$ almost surely. (This is Theorem **??** from class, which I stated but did not prove.)
3. Consider the Binomial model with $N = \infty$, and $0 < d < 1 + r < u$. A perpetual American put is an American put option with no expiry date. That is, a perpetual American put with strike K pays $(K - S_n)^+$ if exercised at time n.
   (a) If the arbitrage free price of a perpetual American put is of the form $V_n = f(S_n)$, then find a finite difference equation for f. Also state what $\lim_{s \to 0} f(s)$ and $\lim_{s \to \infty} f(s)$ should be.
   (b) Consider a perpetual American put option with strike $K = 4$. Assume $u = 2$, $d = 1/2$, $r = 1/4$ and $S_0 = 2^m$ for some $m \in \mathbb{Z}$. Show that the function $f(s) = 4 - s$ if $s \leqslant 2$, and $f(s) = 4/s$ otherwise satisfies the correct finite difference equation and boundary conditions from the previous part. Use this to find the arbitrage free price of this option at time n.
   (c) Find an optimal exercise policy for the perpetual American put in the previous part.

Assignment 12 (assigned 2021-11-17, due 2021-11-24).

1. In the Binomial model we characterized the wealth of self financing portfolios as those for which $X'_{n+1} = \Delta'_n S'_{n+1} + (1+r)(X'_n - \Delta'_n S'_n)$, where S'_n denotes the stock price at time n and Δ'_n is adapted. In the multiple asset model, we characterized trading strategies as those for which $\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}$, where $\Delta_n = (\Delta_n^0, \dots, \Delta_n^d)$ is adapted, and $S_n = (S_n^0, \dots, S_n^d)$ is the vector of asset prices, with S^0 denoting the risk free asset (bank). Now set $d = 1$, $S_n^0 = (1+r)^n$, $S_n^1 = S'_n$. Given a portfolio with wealth process X'_n holding Δ'_n shares of the stock (in the Binomial model), write down a formula for the trading strategy Δ_n in the vector notation from the multiple asset model. Use this to show that a portfolio is self-financing (as we defined for the Binomial model) if and only if the corresponding trading strategy is self-financing (as we defined in the multiple asset model).
2. Let $M \in \mathbb{N}$, $\bar{Q} = \{v \in \mathbb{R}^M \mid v_1 \geqslant 0, \dots, v_M \geqslant 0\}$, $\mathring{Q} = \{v \in \mathbb{R}^M \mid v_1 > 0, \dots, v_M > 0\}$. Let $V \subseteq \mathbb{R}^M$ be a subspace with $\dim(V) = M - 1$. Show that V has a unique unit normal vector in $\mathring{Q}$ if and only if $V \cap \bar{Q} = \{0\}$.
   [I stated, but did not prove this in class. If $\dim(V) < M - 1$, then proving existence of a normal vector in $\mathring{Q}$ is tricky and uses the Hyperplane separation theorem. If $\dim(V) = M - 1$ then the proof is much simper, and is what is asked of you in this question.]
3. Let $\Omega = \{1, \dots, M\}^N$ represent a probability space of N rolls of M-sided dies, and consider the multiple asset model with d stocks and a bank. Let $S = (S^0, \dots, S^d)$ denote the vector process of asset prices with S^0 being the price process of the bank. Fix $\omega' = (\omega_1, \dots, \omega_n)$, and set

$$U = \{(\Delta_n(\omega') \cdot S_{n+1}(\omega', 1), \dots, \Delta_n(\omega') \cdot S_{n+1}(\omega', M)) \mid \Delta_n(\omega') \in \mathbb{R}^{d+1}\},$$
$$V = \{(\Delta_n(\omega') \cdot S_{n+1}(\omega', 1), \dots, \Delta_n(\omega') \cdot S_{n+1}(\omega', M)) \mid \Delta_n(\omega') \cdot S_n(\omega') = 0\}.$$

   (a) Show that U, V are (vector) subspaces of $\mathbb{R}^M$, and $\dim(V) \leqslant d$.
   (b) Suppose the market is arbitrage free. Show that $\dim(V) = M - 1$ if and only if $\dim(U) = M$.
   (c) Must $\dim(V) = \dim(U) - 1$? Prove it, or find a counter example.