Assignment 13 (assigned 2021-11-26, due 2021-12-03).

1. Consider a market which has a money market account with interest rate $r = 5\%$ and three stocks with price processes denoted by S^1, S^2, S^3 respectively. The price of each stock changes according to the roll of a four sided die as follows: $S^i_{n+1} = f_{i,j} S_n$ if $\omega_{n+1} = j$. Here $f_{i,j}$ is the matrix

$$(f_{i,j}) = \begin{pmatrix} 1.15 & 1 & .9 & 1 \\ 1 & 1.1 & .8 & 1 \\ 1 & 1 & 1 & 1.4 \end{pmatrix}$$

   At time $n = 0$ all three stocks are worth \$10. Find the arbitrage free price of an European call option on S^1 with strike price \$10 and maturity $N = 2$. Also find Δ_0 (the position of the replicating portfolio in each of the assets) at time 0.

2. Consider a market which has a money market account with interest rate $r > 0$ and two stocks with price processes denoted by S^1 and S^2 respectively. The price of each stock changes according to the roll of a three sided die: If the die rolls $j \in \{1, \dots, 3\}$ the new price is $f_{i,j} S^i$, for some given factors $f_{i,j} > 0$. Suppose further $f_{i,1} \geqslant f_{i,2} = 1 \geqslant f_{i,3}$ for $i \in \{1, 2\}$. Find a necessary and sufficient condition that the market is complete and arbitrage free.

3. (a) Let $\mathcal{Q} = \{\tilde{\boldsymbol{P}}^\alpha\}$ be the set of all risk neutral measures. For each $\tilde{\boldsymbol{P}}^\alpha \in \mathcal{Q}$, let $\tilde{p}^\alpha$ be its probability mass function. Given $\theta \in [0,1]$, and $\tilde{\boldsymbol{P}}^\alpha, \tilde{\boldsymbol{P}}^\beta \in \mathcal{Q}$ define the measure $\theta \tilde{\boldsymbol{P}}^\alpha + (1-\theta)\tilde{\boldsymbol{P}}^\beta$ to be the probability measure with probability mass function $\theta \tilde{p}^\alpha + (1-\theta)\tilde{p}^\beta$. For every $\theta \in [0,1]$, and $\tilde{\boldsymbol{P}}^\alpha, \tilde{\boldsymbol{P}}^\beta \in \mathcal{Q}$ must $\theta \tilde{\boldsymbol{P}}^\alpha + (1-\theta)\tilde{\boldsymbol{P}}^\beta \in \mathcal{Q}$? (I.e. is the set of all risk neutral measures a convex set?)

   (b) Consider an arbitrage free market with d risky assets and a bank. Consider a new security on this market, that may or may not be replicable. Let $\mathcal{V}_0$ be the set of all possible arbitrage free prices of this security at time 0. Must $\mathcal{V}_0$ either be empty, or be an interval? Prove it, or find a counter example.

4. Let $0 < d < 1 + r < u$, $d < 1 < u$, and consider a market model consisting of a bank with interest rate r, and a stock with price process S_n defined as follows: We start with $S_0 > 0$. At time n, roll a 3 sided die and set $S_{n+1} = u S_n$ if we roll 1, $S_{n+1} = S_n$ if we roll 2 and $S_{n+1} = d S_n$ if we roll 3.

   (a) Is this market complete? Is it arbitrage free? Prove it.

   Suppose we add a second risky asset to this market whose price is given by $V_n = (S_n - S_0)^+$ for all $n \geqslant 1$.

   (b) If N (the total number of periods) is 1, find all $V_0 \in \mathbb{R}$ for which this market is complete and arbitrage free.

   (c) Suppose now $N = 2$ and $ud < 1$, and $V_0 \in \mathbb{R}$ is any one of the values you found in the previous part. Is this market complete? Is it arbitrage free? Prove or disprove your answer.

Assignment 14 (assigned 2021-12-01, due Never).

In light of your final, this homework is not due.

1. Let X_n be a sequence of infinitely many fair coin flips. A bet of B_n dollars at time n pays $B_n X_{n+1}$ dollars at time $n+1$. The bet B_n may be random but has to be adapted (i.e. B_n can only depend on the first n coin flips).

   (a) Let B_n be an adapted process, and consider a gambler that bets B_n dollars at time n. Let M_n denote the cumulative gain/loss of the gambler up to (and including) time n. By convention, we set $M_0 = 0$. Write down a formula for M_n. Is it a martingale?

   A gambler decides to go *double or nothing*. He bets \$1 at time 0. After that, he doubles his bet if he loses and collects his winnings and stops playing if he wins. Let τ be the time he stops playing, B_n his bet at time n and M_n be his cumulative gain/loss up to (and including) time n.

   (b) Is τ a stopping time? Is τ finite almost surely?

   (c) Compute $\boldsymbol{E} M_n$, $\boldsymbol{E} M_\tau$, $\boldsymbol{E} M_\tau^2$

   (d) We've proved Doob's optional sampling theorem when τ is bounded. Does it apply here?

   (e) More generally Doob's optional sampling theorem also applies when there exists $C \in \mathbb{R}$ such that $\boldsymbol{E}\tau < \infty$ and $\boldsymbol{E}_n(\mathbf{1}_{\tau > n} |X_{n+1} - X_n|) \leqslant C$ for all $n \in \mathbb{N}$. Are either of these conditions satisfied here?

2. Consider the Binomial model with interest rate $r \geqslant 0$, maturity $N = \infty$, and $1 + r < u = 1/d$. Fix $m_0, n_0 \in \mathbb{N}$ and define $L = d^{m_0} S_0$ and $U = u^{n_0} S_0$. Let $\tau = \min\{n \in \mathbb{N} \mid S_n \notin (L, U)\}$.

   (a) Suppose $u = 2$, $r = 1/2$, and $A, B \in \mathbb{R}$. An option matures at the random time τ and pays A if $S_\tau = L$ and B if $S_\tau = U$. Find the arbitrage free price of this option at time 0.

   (b) Do the previous part without assuming $u = 2$ and $r = 1/2$.

3. I may add more problems later.