LECTURE NOTES ON DISCRETE TIME FINANCE
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Contents

1. Preface.

These are the slides I used while teaching this course in 2020. I projected them (spaced out) in class, and filled in the proofs by writing over them with a tablet. Both the annotated version of these slides with handwritten proofs, and the compactified un-annotated version can be found on the class website. The LaTeXsource of these slides is also available on git.

2. Syllabus Overview

- Class website and full syllabus: http://www.math.cmu.edu/~gautam/sj/teaching/2020-21/370-dtime-finance
- TA's: Jonghwa Park <jonghwap@andrew.cmu.edu>, Karl Xiao <kzx@andrew.cmu.edu>, Hongyi Zhou <hongyizh@andrew.cmu.edu>.
- Homework Due: Every Wednesday, before class (on Gradescope)
- Midterms: Wed Sep 29, 5th week, and Wed Nov 3rd, 10th week (self proctored, can be taken any time)
- **Zoom lectures:**
  - Please enable video. (It helps me pace lectures).
  - Mute your mic when you're not speaking. Use headphones if possible. Consent to be recorded.
  - If I get disconnected, check your email for instructions.
- **Homework:**
  - Good quality scans please! Use a scanning app, and not simply take photos. (I use Adobe Scan.)
  - 20% penalty if turned in within an hour of the deadline. 100% penalty after that.
  - Two homework assignments can be turned in 24h late without penalty.
  - Bottom 2 homework scores are dropped from your grade (personal emergencies, other deadlines, etc.).
  - Collaboration is encouraged. Homework is not a test – ensure you learn from doing the homework.
  - You must write solutions independently, and can only turn in solutions you fully understand.
- **Exams:**
  - Can be taken at any time on the exam day. Open book. Use of internet allowed.
  - Collaboration is forbidden. You may not seek or receive assistance from other people. (Can search forums; but may not post.)
  - Self proctored: Zoom call. Record yourself, and your screen to the cloud.
  - Share the recording link; also download a copy and upload it to the designated location immediately after turning in your exam.
- **Academic Integrity**
  - Zero tolerance for violations (automatic **R**).

- ▷ Violations include:
  - Not writing up solutions independently and/or plagiarizing solutions
  - Turning in solutions you do not understand.
  - Seeking, receiving or providing assistance during an exam.
  - Discussing the exam on the exam day (24h). Even if you have finished the exam, others may be taking it.
- ▷ All violations will be reported to the university, and they may impose additional penalties.
- **Grading:** 30% homework, 20% each midterm, 30% final.

3. Replication, and Arbitrage Free Pricing

- Start with a *financial market* consisting of traded assets (stocks, bonds, money market, options, etc.)
- We model the price of these assets through random variables (stochastic processes).
- **No Arbitrage Assumption:**
  - ▷ In order to make money, you have to take risk. (Can't make something out of nothing.)
  - ▷ There doesn't exist a trading strategy with $X_0 = 0$, $X_n \geqslant 0$ and $\boldsymbol{P}(X_n > 0) > 0$.
- Now consider a non-traded asset Y (e.g. an option). How do you price it?
- *Arbitrage free price:* V_0 is the arbitrage free price of Y, if given the opportunity to trade Y at price V_0, the market remains arbitrage free.
- How do you compute the arbitrage free price? Replication:
  - ▷ Say the non-traded asset pays V_N at time N (e.g. call options).
  - ▷ Say you can *replicate* the payoff through a trading strategy $X_0, \ldots, X_N = V_N$ (using only traded assets).
  - ▷ Then the arbitrage free price is uniquely determined, and must be X_0.

Question 3.1. *Is the arbitrage free price always unique?*

Theorem 3.2. *The arbitrage free price is* unique *if and only if there is a replicating strategy! In this case, the arbitrage free price is exactly the initial capital of the replicating strategy.*

Proof. We already proved that if a replicating strategy exists then the arbitrage free price is unique. The other direction is harder, and will be done later. ☐

Question 3.3. *If a replicating strategy exists, must it be unique?*

Question 3.4. *Consider a financial market with a money market account with interest rate r, and a stock. Let $K > 0$. A* forward contract *requires the holder to buy the stock at price K at maturity time N. What is the arbitrage free price at time 0?*

4. Binomial model (one period)

Say we have access to a money market account with interest rate r. The *binomial model* dictates that the stock price varies as follows. Let $p \in (0,1)$, $q = 1 - p$, $0 < d < u$ (up and down factors). Flip a coin that lands heads with probability p, and tails with probability q. When the coin lands heads, the stock price changes by the factor u, and when it lands tails it changes by the factor d.

Question 4.1. *When is there arbitrage in this market?*

Question 4.2. *If a security pays V_1 at time 1, what is the arbitrage free price at time 0. (V_1 can depend on whether the coin flip is heads or tails).*

Question 4.3. *What's an N period version of this model? Do we have the same formulae?*

5. A quick introduction to probability

This is just a quick reminder, and specific to our situation (coin toss spaces). You should have already taken a probability course, or be co-enrolled in one. The only thing we will cover in any detail is conditional expectation.

Let $N \in \mathbb{N}$ be large (typically the maturity time of financial securities).

Definition 5.1. The *sample space* is the set $\Omega = \{(\omega_1, \dots, \omega_N) \mid \text{each } \omega_i \text{ represents the outcome of a coin toss (or die roll).}\}$

- ▷ E.g. $\omega_i \in \{H, T\}$, or $\omega_i \in \{\pm 1\}$.
- ▷ Coins / dice don't have to be identical: Pick M_1, M_2, …, $\in \mathbb{N}$, and can require $\omega_i \in \{1, \dots, M_i\}$.
- ▷ Usually in probability the *sample space* is simply a set; however, for our purposes it is more convenient to consider "coin toss spaces" as we defined above.

Definition 5.2. A *sample point* is a point $\omega = (\omega_1, \dots, \omega_N) \in \Omega \in \Omega$.

Definition 5.3. A *probability mass function* is a function $p \colon \Omega \to [0, 1]$ such that $\sum_{\omega \in \Omega} p(\omega) = 1$.

Definition 5.4. An event is a subset of Ω. Define $\boldsymbol{P}(A) = \sum_{\omega \in A} p(\omega)$.

5.1. Random Variables and Independence.

Definition 5.5. A *random variable* is a function $X \colon \Omega \to \mathbb{R}$.

Question 5.6. *What is the random variable corresponding to the outcome of the n^{th} coin toss?*

Definition 5.7. The *expectation* of a random variable X is $\boldsymbol{E}X = \sum X(\omega)p(\omega)$.

Remark 5.8. Note if $\text{Range}(X) = \{x_1, \dots, x_n\}$, then $\boldsymbol{E}X = \sum X(\omega)p(\omega) = \sum_1^n x_i \boldsymbol{P}(X = x_i)$.

Definition 5.9. The *variance* of a random variable is $\text{Var}(X) = \boldsymbol{E}(X - \boldsymbol{E}X)^2$.

Remark 5.10. Note $\text{Var}(X) = \boldsymbol{E}X^2 - (\boldsymbol{E}X)^2$.

Definition 5.11. Two events are independent if $\boldsymbol{P}(A \cap B) = \boldsymbol{P}(A)\boldsymbol{P}(B)$.

Definition 5.12. The events A_1, …, A_n are independent if for any sub-collection $A_{i_1}, \dots, A_{i_k}$ we have

$$\boldsymbol{P}(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = \boldsymbol{P}(A_{i_1})\boldsymbol{P}(A_{i_2}) \cdots \boldsymbol{P}(A_{i_k}) \,.$$

Remark 5.13. When $n > 2$, it is not enough to only require $\boldsymbol{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \boldsymbol{P}(A_1)\boldsymbol{P}(A_2)\cdots\boldsymbol{P}(A_n)$

Definition 5.14. Two random variables are independent if $\boldsymbol{P}(X = x,\ Y = y) = \boldsymbol{P}(X = x)\boldsymbol{P}(Y = y)$ for all $x, y \in \mathbb{R}$.

Definition 5.15. The random variables $X_1, \ldots, X_n$ are independent if for all $x_1, \ldots, x_n \in \mathbb{R}$ we have

$$\boldsymbol{P}(X_1 = x_1,\ X_2 = x_2, \ldots, X_n = x_n) = \boldsymbol{P}(X_1 = x_1)\boldsymbol{P}(X_2 = x_2)\cdots\boldsymbol{P}(X_n = x_n)\,.$$

Remark 5.16. Independent random variables are uncorrelated, but not vice versa.

Proposition 5.17. *The coin tosses in our setup are all independent, if and only if, there exists functions $p_1, \ldots, p_N$ such that*

$$p(\omega) = p_1(\omega_1)p_2(\omega_2)\cdots p_N(\omega_N)\,.$$

5.2. **Filtrations.**

- Let $N \in \mathbb{N}$, $d_1, \ldots, d_N \in \mathbb{N}$, $\Omega = \{1, \ldots, d_1\} \times \{1, \ldots, d_n\} \times \cdots \times \{1, \ldots, d_N\}$.
- That is $\Omega = \{\omega \mid \omega = (\omega_1, \ldots, \omega_N),\ \omega_i \in \{1, \ldots, d_i\}\}$.
- $d_n = 2$ for all n corresponds to flipping a two sided coin at every time step.

Definition 5.18. We define a *filtration* on Ω as follows:

▷ $\mathcal{F}_0 = \{\emptyset, \Omega\}$.
▷ $\mathcal{F}_1 =$ all events that can be described by only the first coin toss (die roll). E.g. $A = \{\omega \mid \omega_1 = H\} \in \mathcal{F}_1$.
▷ $\mathcal{F}_n =$ all events that can be described by only the first n coin tosses.

More precisely, given $\omega = (\omega_1, \ldots, \omega_N) \in \Omega$ and $n \in \{0, \ldots, N\}$ define

$$\Pi_n(\omega) = \{\omega' \in \Omega \mid \omega' = (\omega'_1, \ldots, \omega'_N) \text{ and } \omega'_i = \omega_i \text{ for all } i \leqslant n\}\,.$$

Now $\mathcal{F}_n$ is defined by $\mathcal{F}_n \stackrel{\text{def}}{=} \Big\{A \subseteq \Omega \;\Big|\; A = \bigcup_{i=1}^{k} \Pi_n(\omega^i),\ \omega^1, \ldots, \omega^k \in \Omega\Big\}$

Remark 5.19. Note $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_N = \mathcal{P}(\Omega)$.

Question 5.20. *Let $\Omega = \{H, T\}^3 \cong \{1, 2\}^3$. What are $\mathcal{F}_0$, ..., $\mathcal{F}_3$?*

Definition 5.21. Let $n \in \{0, \dots, N\}$. We say a random variable X is $\mathcal{F}_n$-measurable if $X(\omega)$ only depends on ω_1, ..., ω_n.

- ▷ Equivalently, for any $B \subseteq \mathbb{R}$, the event $\{X \in B\} \in \mathcal{F}_n$.
- ▷ Equivalently, if $\omega' \in \Pi_n(\omega)$ then $X(\omega') = X(\omega)$.

Question 5.22. *Let $X(\omega) \stackrel{\text{def}}{=} \omega_1 - 10\omega_2$. For what n is $\mathcal{F}_n$-measurable?*

5.3. Conditional expectation.

Definition 5.23. Let X be a random variable, and $n \leqslant N$. We define $\boldsymbol{E}(X \mid \mathcal{F}_n) = \boldsymbol{E}_n X$ to be the *random variable* given by

$$\boldsymbol{E}_n X(\omega) = \frac{\displaystyle\sum_{\omega' \in \Pi_n(\omega)} p(\omega') X(\omega')}{\displaystyle\sum_{\omega' \in \Pi_n(\omega)} p(\omega')}, \qquad \text{where} \qquad \Pi_n(\omega) = \{\omega' \in \Omega \mid \omega'_1 = \omega_1, \ \dots, \omega'_n = \omega_n\}$$

Remark 5.24. $\boldsymbol{E}_n X$ is the "best approximation" of X given only the first n coin tosses.

Remark 5.25. The above formula does not generalize well to infinite probability spaces. We will develop a definition that does generalize; after we have that definition we will never ever ever use this formula.

Proposition 5.26. *The conditional expectation $\boldsymbol{E}_n X$ defined by the above formula satisfies the following two properties:*

(1) *$\boldsymbol{E}_n X$ is an $\mathcal{F}_n$-measurable* random variable.

(2) *For every $A \in \mathcal{F}_n$, $\displaystyle\sum_{\omega \in A} \boldsymbol{E}_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$.*

Proof of (2):

(1) For any $\omega \in \Omega$, $\displaystyle\sum_{\omega' \in \Pi_n(\omega)} \boldsymbol{E}_n X(\omega')p(\omega') = \sum_{\omega' \in \Pi_n(\omega)} X(\omega')p(\omega')$

(2) For any $A \subseteq \mathcal{F}_n$, then there exist $\omega^1, \dots, \omega^k \in \Omega$ such that A is the *disjoint union* of $\Pi_n(\omega^1), \dots, \Pi_n(\omega^k)$.

(3) Hence $\displaystyle\sum_{\omega \in A} \boldsymbol{E}_n X(\omega)p(\omega) = \sum_{i=1}^{k} \sum_{\omega \in \Pi_n(\omega^i)} \boldsymbol{E}_n X(\omega)p(\omega) = \sum_{i=1}^{k} \sum_{\omega \in \Pi_n(\omega^i)} X(\omega)p(\omega) = \sum_{\omega \in A} X(\omega)p(\omega)$.

Proposition 5.27 (Uniqueness). *If Y and Z are two $\mathcal{F}_n$-measurable random variables such that $\sum_{\omega \in A} Y(\omega)p(\omega) = \sum_{\omega \in A} Z(\omega)p(\omega)$ for every $A \in \mathcal{F}_n$, then we must have $\boldsymbol{P}(Y = Z) = 1$.*

Definition 5.28. Let X be a random variable, and $n \leqslant N$. We define the *conditional expectation of X given $\mathcal{F}_n$*, denoted by $\boldsymbol{E}_n X$, or $\boldsymbol{E}(X \mid \mathcal{F}_n)$, to be the unique *random variable* such that:

(1) $\boldsymbol{E}_n X$ is a $\mathcal{F}_n$-measurable random variable.
(2) For every $A \subseteq \mathcal{F}_n$, we have $\sum_{\omega \in A} \boldsymbol{E}_n X(\omega)p(\omega) = \sum_{\omega \in A} X(\omega)p(\omega)$.

Remark 5.29. This is the definition that generalizes to the continuous case. All properties we develop on conditional expectations will only use the above definition, and not the explicit formula.

Example 5.30. Let S_n be the stock price in the binomial model after n periods. Compute $E_1 S_3$, $E_2 S_3$.

Theorem 5.31. *(1) If X, Y are two random variables and $\alpha \in \mathbb{R}$, then $\boldsymbol{E}_n(X + \alpha Y) = \boldsymbol{E}_n X + \alpha \boldsymbol{E}_n Y$. (On homework).*
(2) (Tower property) *If $m \leqslant n$, then $\boldsymbol{E}_m(\boldsymbol{E}_n X) = \boldsymbol{E}_m X$.*
(3) If X is $\mathcal{F}_n$ measurable, and Y is any random variable, then $\boldsymbol{E}_n(XY) = X\boldsymbol{E}_n Y$.

Theorem 5.32. *If X is independent of $\mathcal{F}_n$ then $\boldsymbol{E}_n X = \boldsymbol{E} X$.*

Theorem 5.33 (Independence lemma)**.** *If X is independent of $\mathcal{F}_n$ and Y is $\mathcal{F}_n$-measurable, and $f : \mathbb{R} \to \mathbb{R}$ is a function then*

$$\boldsymbol{E}_n f(X, Y) = \sum_{i=1}^{m} f(x_i, Y) \boldsymbol{P}(X = x_i)\,, \qquad \textit{where } \{x_1, \dots, x_m\} = X(\Omega)\,.$$

5.4. **Martingales.**

Definition 5.34. A *stochastic process* is a collection of random variables X_0, X_1, …, X_N.

Definition 5.35. A stochastic process is *adapted* if X_n is $\mathcal{F}_n$-measurable for all n. (Non-anticipating.)

Question 5.36. *Is $X_n(\omega) = \sum_{i \leqslant n} \omega_i$ adapted?*

Question 5.37. *Is $X_n(\omega) = \omega_n$ adapted? Is $X_n(\omega) = 15$ adapted? Is $X_n(\omega) = \omega_{15}$ adapted? Is $X_n(\omega) = \omega_{N-i}$ adapted?*

Remark 5.38*.* We will always model the price of assets by *adapted* processes. We will also only consider trading strategies which are adapted.

Example 5.39 (Money market)*.* Let $Y_0 = Y_0(\omega) = a \in \mathbb{R}$. Define $Y_{n+1} = (1 + r)Y_n$. (Here r is the interest rate.)

Example 5.40*.* Suppose $\Omega = \{\pm 1\}^N \cong \{H, T\}^N \cong \{1, 2\}^N$. Let $S_0 = a \in \mathbb{R}$. Define $S_{n+1}(\omega) = \begin{cases} uS_n(\omega) & \omega_{n+1} = 1\,, \\ dS_n(\omega) & \omega_{n+1} = -1\,. \end{cases}$
Is S_n adapted? (Used to model stock price in the multi-period Binomial model.)

Definition 5.41. We say an adapted process M_n is a martingale if $\boldsymbol{E}_n M_{n+1} = M_n$. (Recall $\boldsymbol{E}_n Y = \boldsymbol{E}(Y \mid \mathcal{F}_n)$.)

Remark 5.42*.* Intuition: A martingale is a "fair game".

Example 5.43 (Unbiased random walk)*.* If $X_1, \dots, X_N$ are i.i.d. and mean zero, then $S_n = \sum_{k=1}^{n} X_k$ is a martingale.

Example 5.44 (Drawing balls without replacement). Red or Blue balls are drawn from a container *without replacement*. The container has 2 red and 2 balls initially. You win \$1 if the ball is blue, and lose \$1 if the ball is red. Is the process of your winnings a martingale?

Question 5.45. *If M is a martingale, and $m \leqslant n$, is $\boldsymbol{E}_m M_n = M_m$?*

Question 5.46. *If M is a martingale does $\boldsymbol{E} M_n$ change with n?*

Question 5.47. *Conversely, if $\boldsymbol{E} M_n$ is constant, is M a martingale?*

Question 5.48. *If M is a martingale, must $M_{n+1} - M_n$ be independent of $\mathcal{F}_n$?*

Question 5.49. *If $M_{n+1} - M_n$ is mean 0 and independent of $\mathcal{F}_n$, then is M is a martingale?*

Question 5.50. *Let ξ_n be a martingale with $\boldsymbol{E}\xi_1 = 0$. Let Δ_n be an adapted process, $X_0 \in \mathbb{R}$ and define $X_{n+1} = X_n + \Delta_n \xi_{n+1}$. Is X a martingale?*

Remark 5.51. Think of ξ_n as the outcome of a fair game being played. You decide to bet on this game. Let Δ_n be your bet at time n; your return from this bet is $\Delta_n \xi_{n+1}$, and thus your cumulative return at time $n+1$ is $X_{n+1} = X_n + \Delta_n \xi_{n+1}$.

5.5. **Change of measure.**

Example 5.52. Consider i.i.d. coin tosses with $\boldsymbol{P}(\omega_n = 1) = p_1$ and $\boldsymbol{P}(\omega_n = -1) = q_1 = 1 - p_1$. Let $u, d > 0$, $r > -1$. Let $S_{n+1}(\omega) = u S_n(\omega)$ if $\omega_{n+1} = 1$, and $S_{n+1}(\omega) = d S_n(\omega)$ if $\omega_{n+1} = -1$. Let $D_n = (1+r)^{-n}$ be the "discount factor".

Suppose we now invented a new "risk neutral" coin that comes up heads with probability $\tilde{p}_1$ and tails with probability $\tilde{q}_1 = 1 - \tilde{p}_1$. Let $\tilde{\boldsymbol{P}}$, $\tilde{\boldsymbol{E}}_n$ etc. denote the probability and conditional expectation with respect to the new "risk neutral" coin. Find $\tilde{p}_1$ so that $D_n S_n$ is a $\tilde{\boldsymbol{P}}$ martingale.

Theorem 5.53. *Consider a market where S_n above models a stock price, and r is the interest rate with $0 < d < 1 + r < u$. The coins land heads and tails with probability p_1 and q_1 respectively. If you have a derivative security that pays V_N at time N, then the arbitrage* *free price of this security at time $n \leqslant N$ is given by*

$$V_n = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n D_N V_N = (1+r)^{n-N} \tilde{\boldsymbol{E}}_n V_N \,.$$

Remark 5.54. Even though the stock price changes according to a coin that flips heads with probability p_1, the arbitrage free price is computed using conditional expectations using the *risk neutral probability*. So when computing $\tilde{\boldsymbol{E}}_n V_N$, we use our new invented "risk neutral" coin that flips heads with probability $\tilde{p}_1$ and tails with probability $\tilde{q}_1$.

- Let $p \colon \Omega \to [0,1]$ be a probability mass function on Ω, and $\boldsymbol{P}(A) = \sum_{\omega \in A} p(\omega)$ be the probability measure.
- Let $\tilde{p} \colon \Omega \to [0,1]$ be another probability mass function, and define a second probability measure $\tilde{\boldsymbol{P}}$ by $\tilde{\boldsymbol{P}}(A) = \sum_{\omega \in A} \tilde{p}(\omega)$.

Definition 5.55. We say $\boldsymbol{P}$ and $\tilde{\boldsymbol{P}}$ are equivalent if for every $A \in \mathcal{F}_N$, $\boldsymbol{P}(A) = 0$ if and only if $\tilde{\boldsymbol{P}}(A) = 0$.

Remark 5.56. When Ω is finite, $\boldsymbol{P}$ and $\tilde{\boldsymbol{P}}$ are equivalent if and only if we have $p(\omega) = 0 \iff \tilde{p}(\omega) = 0$ for all $\omega \in \Omega$.

We let $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{E}}_n$ denote the expectation and conditional expectations with respect to $\tilde{\boldsymbol{P}}$ respectively.

Work out Example 5.52

Example 5.57. Let Ω be the sample space corresponding to N i.i.d. fair coins (heads is 1, tails is -1). Let $a \in \mathbb{R}$ and define $X_{n+1}(\omega) = X_n(\omega) + \omega_{n+1} + a$. For what a is there an equivalent measure $\tilde{\boldsymbol{P}}$ such that X is a martingale?

6. The multi-period binomial model

6.1. Risk Neutral Pricing.

- In the multi-period binomial model we assume $\Omega = \{\pm 1\}^N$ corresponds to a probability space with N i.i.d. coins.
- Let $u, d > 0$, $S_0 > 0$, and define $S_{n+1} = \begin{cases} uS_n & \omega_{n+1} = 1\,, \\ dS_n & \omega_{n+1} = -1\,. \end{cases}$

- u and d are called the up and down factors respectively.
- Without loss, can assume $d < u$.
- Always assume no coins are deterministic: $p_1 = \boldsymbol{P}(\omega_n = 1) > 0$ and $q_1 = 1 - p_1 = \boldsymbol{P}(\omega_n = -1) > 0$.
- We have access to a bank with interest rate $r > -1$.
- $D_n = (1+r)^{-n}$ be the discount factor (\$1 at time n is worth \$$D_n$ at time 0.)

Theorem 6.1. *There exists a (unique) equivalent measure $\tilde{\boldsymbol{P}}$ under which process $D_n S_n$ is a martingale* if and only if $d < 1 + r < u$. *In this case $\tilde{\boldsymbol{P}}$ is the probability measure obtained by tossing N i.i.d. coins with*

$$\tilde{\boldsymbol{P}}(\omega_n = 1) = \tilde{p}_1 = \frac{1+r-d}{u-d}, \qquad \tilde{\boldsymbol{P}}(\omega_n = -1) = \tilde{q}_1 = \frac{u-(1+r)}{u-d}.$$

Definition 6.2. An equivalent measure $\tilde{\boldsymbol{P}}$ under which $D_n S_n$ is a martingale is called the *risk neutral measure.*

Remark 6.3. If there are more than one risky assets, $S^1, \dots, S^k$, then we require $D_n S_n^1, \dots, D_n S_n^k$ to all be martingales under the risk neutral measure $\tilde{\boldsymbol{P}}$.

Remark 6.4. The *Risk Neutral Pricing Formula* says that any security with payoff V_N at time N has arbitrage free price $V_n = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n(D_N V_N)$ at time n.

Example 6.5. Consider two markets in the Binomial model setup with the same u, d, r. In the first market the coin flip heads with probability 99%. In the second the coin flips heads with probability 90%. Are the price of call options in these two markets the same?

- Consider an investor that starts with X_0 wealth, which he divides between cash and the stock.
- If he has Δ_0 shares of stock at time 0, then $X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$.
- We allow the investor to trade at time 1 and hold Δ_1 shares.
- Δ_1 may be random, *but must be $\mathcal{F}_1$-measurable.*

- Continuing further, we see $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$.
- *Both X and Δ are adapted processes.*

Definition 6.6. A *self-financing portfolio* is a portfolio whose wealth evolves according to

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)\,,$$

for some adapted process Δ_n.

Theorem 6.7. *Let $d < 1 + r < u$, and $\tilde{\boldsymbol{P}}$ be the risk neutral measure, and X_n represent the wealth of a portfolio at time n. The portfolio is self-financing portfolio if and only if the discounted wealth $D_n X_n$ is a martingale under $\tilde{\boldsymbol{P}}$.*

Remark 6.8. The only thing we will use in this proof is that $D_n S_n$ is a martingale under $\tilde{\boldsymbol{P}}$. The interest rate r can be a *random* adapted process. It is also not special to the binomial model – it works for any model for which there is a risk neutral measure.

Before proving Theorem 6.7, we consider a few consequences:

Theorem 6.9. *The multi-period binomial model is arbitrage free if and only if $d < 1 + r < u$.*

Definition 6.10. We say the market is arbitrage free if for any self financing portfolio with wealth process X, we have: $X_0 = 0$ and $X_N \geqslant 0$ implies $X_N = 0$ almost surely.

Remark 6.11. The first fundamental theorem of asset pricing states that a risk neutral measure exists if and only if the market is arbitrage free. (We will prove this in more generality later.)

Theorem 6.12 (Risk Neutral Pricing Formula)**.** *Let $d < 1 + r < u$, and V_N be an $\mathcal{F}_N$ measurable random variable. Consider a security that pays V_N at maturity time N. For any $n \leqslant N$, the arbitrage free price of this security is given by*

$$V_n = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n(D_N V_N)\,.$$

Remark 6.13. The replicating strategy can be found by backward induction. Let $\omega = (\omega', \omega_{n+1}, \omega'')$. Then

$$\Delta_n(\omega) = \frac{V_{n+1}(\omega', 1, \omega'') - V_{n+1}(\omega', -1, \omega'')}{(u-d)S_n(\omega)} = \frac{V_{n+1}(\omega', 1) - V_{n+1}(\omega', -1)}{(u-d)S_n(\omega)}$$

Proof of Theorem 6.7 part 1. Suppose X_n is the wealth of a self-financing portfolio. Need to show $D_n X_n$ is a martingale under $\tilde{\boldsymbol{P}}$.
Proof of Theorem 6.7 part 2. Suppose $D_n X_n$ is a martingale under $\tilde{\boldsymbol{P}}$. Need to show X_n is the wealth of a self-financing portfolio.

6.2. State processes.

Question 6.14. *Consider the N-period binomial model, and a security with payoff V_N. Let X_n be the arbitrage free price at time $n \leqslant N$, and Δ_n be the number of shares in the replicating portfolio. What is an algorithm to find X_n, Δ_n for all $n \leqslant N$? How much is the computational time?*

Theorem 6.15. *Suppose a security pays $V_N = g(S_N)$ at maturity N for some (non-random) function g. Then the arbitrage free price at time $n \leqslant N$ is given by $V_n = f_n(S_n)$, where:*

(1) $f_N(x) = g(x)$ *for* $x \in \operatorname{Range}(S_N)$.
(2) $f_n(x) = \dfrac{1}{1+r}(\tilde{p} f_{n+1}(ux) + \tilde{q} f_{n+1}(dx))$ *for* $x \in \operatorname{Range}(S_n)$.

Remark 6.16. Reduces the computational time from $O(2^N)$ to $O(\sum_0^N |\operatorname{Range}(S_n)|) = O(N^2)$ for the Binomial model.

Remark 6.17. Can solve this to get $f_n(x) = \dfrac{1}{(1+r)^{N-n}} \displaystyle\sum_{k=0}^{N-n} \binom{N-n}{k} \tilde{p}^k \tilde{q}^{N-n-k} f_N(x u^k d^{N-n-k})$

Question 6.18. *How do we handle other securities? E.g. Asian options (of the form $g(\sum_0^N S_k)$)?*

Example 6.19 (Knockout options). An *up and out call* option with strike K and barrier U and maturity N gives the holder the option (not obligation) to buy the stock at price K at maturity time N, *provided* the stock price has never exceeded the barrier price U. If the stock price exceeds the barrier U before maturity, the option is worthless. Find an efficient algorithm to price this option.

Definition 6.20. We say a d-dimensional process $Y = (Y^1, \dots, Y^d)$ process is a *state process* if for any security with maturity $m \leqslant N$, and payoff of the form $V_m = f_m(Y_m)$ for some (non-random) function f_m, the arbitrage free price must also be of the form $V_n = f_n(Y_n)$ for some (non-random) function f_n.

Remark 6.21. For state processes given f_N, we typically find f_n by backward induction. The number of computations at time n is of order $\operatorname{Range}(Y_n)$.

Remark 6.22. The fact that S_n is Markov (under $\tilde{\boldsymbol{P}}$) implies that it is a state process.

Theorem 6.23. *Let $Y = (Y^1, \dots, Y^d)$ be a d-dimensional process. Suppose we can find functions $g_1, \dots, g_N$ such that $Y_{n+1}(\omega) = g_{n+1}(Y_n(\omega), \omega_{n+1})$. Then Y is a state process.*

Question 6.24. *Is $Y_n = S_n$ a state process?*

Question 6.25. *Is $Y_n = \max_{k \leqslant n} S_n$ a state process?*

Question 6.26. *Is $Y_n = (S_n, \max_{k \leqslant n} S_n)$ a state process?*

Question 6.27. *Let $A_n = \sum_0^n S_k$. Is A_n a state process?*

Question 6.28. *Is $Y_n = (S_n, A_n)$ a state process?*

6.3. **Options with random maturity.** Consider the N period binomial model with $0 < d < 1 + r < u$.

Example 6.29 (Up-and-rebate option). Let $A, U > 0$. The up-and-rebate option pays the face value A at the first time the stock price exceeds U (up to maturity time N), and nothing otherwise. Explicitly, let $\tau = \min\{n \leqslant N \mid S_n \geqslant U\}$, and let $\sigma = \tau \wedge N$. The up-and-rebate options pays $A\mathbf{1}_{\tau \leqslant N}$ at the random time σ.

Remark 6.30. By convention $\min \emptyset = \infty$.

Definition 6.31. We say a random variable τ is a *stopping time* if:

(1) $\tau \colon \Omega \to \{0, \dots, N\} \cup \infty$
(2) For all $n \leqslant N$, the event $\{\tau \leqslant n\} \in \mathcal{F}_n$.

Remark 6.32. We say τ is a finite stopping time if $\tau < \infty$ almost surely.

Remark 6.33. The second condition above is equivalent to requiring $\{\tau = n\} \in \mathcal{F}_n$ for all n.

Question 6.34. *Is $\tau = 5$ a stopping time?*

Question 6.35. *Is the first time the stock price hits U a stopping time?*

Question 6.36. *Is the last time the stock price hits U a stopping time?*

Question 6.37. *If σ and τ are stopping times, is $\sigma \wedge \tau$ a stopping time? How about $\sigma \vee \tau$?*

- Let G be an adapted process, and σ be a *finite* stopping time.
- Note $G_\sigma = \sum_{n=}^{N} G_n \mathbf{1}_{\sigma = n}$.
- Let $(X_0, (\Delta_n))$ be a self-financing portfolio, and X_n at time n be the wealth of this portfolio at time n.

Definition 6.38. Consider a derivative security that pays G_σ at the random time σ. A self-financing portfolio with wealth process X is a replicating strategy if $X_\sigma = G_\sigma$.

Remark 6.39. If a replicating strategy exists, then at any time before σ, the wealth of the replicating strategy must equal the arbitrage free price V. That is, $\mathbf{1}_{\{n \leqslant \sigma\}} X_n = \mathbf{1}_{\{n \leqslant \sigma\}} V_n$.

Theorem 6.40. *The security with payoff G_σ (at the stopping time σ) can be replicated. The arbitrary free price is given by*

$$V_n \mathbf{1}_{\{\sigma \geqslant n\}} = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n (D_\sigma G_\sigma \mathbf{1}_{\{\sigma \geqslant n\}})$$

Remark 6.41. The only thing required for the proof of Theorem 6.40 is the fact that X_n is the wealth of a self-financing portfolio if and only if $D_n X_n$ is a $\tilde{\boldsymbol{P}}$ martingale.

Proposition 6.42. *The wealth of the replicating portfolio (at times before σ) is uniquely determined by the recurrence relations:*

$$X_N \mathbf{1}_{\{\sigma = N\}} = G_N \mathbf{1}_{\{\sigma = N\}}$$

$$X_n \mathbf{1}_{\{\sigma \geqslant n\}} = G_n \mathbf{1}_{\{\sigma = n\}} + \frac{1}{1+r} \mathbf{1}_{\{\sigma > n\}} \tilde{\boldsymbol{E}}_n X_{n+1} \,.$$

If we write $\omega = (\omega', \omega_{n+1}, \omega'')$ with $\omega' = (\omega_1, \dots, \omega_n)$, then we know in the Binomial model we have

$$\tilde{\boldsymbol{E}}_n X_{n+1}(\omega) = \tilde{\boldsymbol{E}}_n X_{n+1}(\omega') = \tilde{p} X_{n+1}(\omega', 1) + \tilde{q} X_{n+1}(\omega', -1) \,.$$

As before, we will use state processes to find practical algorithms to price securities.

Example 6.43. Let $A, U > 0$. The up-and-rebate option pays the face value A at the first time the stock price exceeds U (up to maturity time N), and nothing otherwise. Find an efficient way to compute the arbitrage free price of this option.

Proposition 6.44. *Let $Y = (Y^1, \dots, Y^d)$ be a d-dimensional process such that for every n we have $Y_{n+1}(\omega) = h_{n+1}(Y_n(\omega), \omega_{n+1})$ for some deterministic function h_{n+1}. Let $A_1, \dots, A_N \subseteq \mathbb{R}^d$, with $A_N = \mathbb{R}^d$, and define the stopping time σ by*

$$\sigma = \min\{n \in \{0, \dots, N\} \mid Y_n \in A_n\} \,.$$

Let $g_0, \dots g_N$ be N deterministic functions on $\mathbb{R}^d$, and consider a security that pays $G_\sigma = g_\sigma(Y_\sigma)$. The arbitrage free price of this security is of the form $V_n \mathbf{1}_{\{\sigma \geqslant n\}} = f_n(Y_n) \mathbf{1}_{\{\sigma \geqslant n\}}$. The functions f_n satisfy the recurrence relation

$$f_N(y) = g_N(y)$$

$$f_n(y) = \mathbf{1}_{\{y \in A_n\}} g_n(y) + \frac{\mathbf{1}_{\{y \notin A_n\}}}{1+r} \Big(\tilde{p} f_{n+1}(h_{n+1}(y, 1)) + \tilde{q} f_{n+1}(h_{n+1}(y, -1)) \Big)$$

6.4. **Optional Sampling.**

Theorem 6.45 (Doob's optional sampling theorem). *Let τ be a bounded stopping time and M be a martingale. Then $\boldsymbol{E}_n M_\tau = M_{\tau \wedge n}$.*

Remark 6.46. When dealing with finitely many coin tosses ($N < \infty$), *bounded stopping times* are the same as *finite stopping times*. When dealing with infinitely many coin tosses, the two notions are different.

Remark 6.47. When $N = \infty$ and τ is not bounded, the optional sampling theorem need not hold (see the Gamblers ruin example below). However, the optional sampling theorem still holds under either of the following assumptions:

(1) If $\tau < \infty$ almost surely, and $M_{\tau \wedge k}$ is uniformly bounded.
(2) If $\boldsymbol{E}\tau < \infty$ and the increments $M_{k+1} - M_k$ are uniformly bounded.

Corollary 6.48. *If M is a martingale and τ is a bounded stopping time, then $\boldsymbol{E} M_\tau = \boldsymbol{E} M_0$.*

Proof of Theorem 6.45

Consider a market with a few risky assets and a bank.

Proposition 6.49. *Suppose a market admits a risk neutral measure. If X is the wealth of a self-financing portfolio and τ is a bounded stopping time such that $X_0 = 0$, and $X_\tau \geqslant 0$, then $X_\tau = 0$. That is, there cant be an arbitrage opportunity at any bounded stopping time.*

Question 6.50 (Gamblers ruin). *Suppose $N = \infty$. Let ξ_n be i.i.d. random variables with mean 0, and let $X_n = \sum_1^n \xi_k$. Let $\tau = \min\{n \mid X_n = 1\}$. (It is known that $\tau < \infty$ almost surely.) What is $\boldsymbol{E} X_\tau$? What is $\lim_{N \to \infty} \boldsymbol{E} X_{\tau \wedge N}$?*

6.5. **American Options.**

An American option is an option that can be exercised at any time chosen by the holder.

Definition 6.51. Let $G_0, G_1, \dots, G_N$ be an adapted process. An *American option* with *intrinsic value* G is a security that pays G_σ at any finite stopping time σ chosen by the holder.

Example 6.52. An *American put* with strike K is an American option with intrinsic value $(K - S_n)^+$.

Question 6.53. *How do we price an American option? How do we decide when to exercise it? What does it mean to replicate it?*

Strategy I: Let σ be a finite stopping time, and consider an option with (random) maturity time σ and payoff G_σ. Let V_0^σ denote the arbitrage free price of this option. The arbitrage free price of the American option *should be* $V_0 = \max_\sigma V_0^\sigma$, where the maximum is taken over all *finite* stopping times σ.

Definition 6.54. The *optimal exercise time* is a stopping time σ^* that maximizes $V_0^{\sigma^*}$ over all finite stopping times.

Definition 6.55. An optimal exercise time σ^* is called *minimal* if for every optimal exercise time τ^* we have $\sigma^* \leqslant \tau^*$.

Remark 6.56. The optimal exercise time need not be unique. (The *minimal* optimal exercise time is certainly unique.)

Strategy II: Replication. Suppose we have sold an American option with intrinsic value G to an investor. Using that, we hedge our position by investing in the market/bank, and let X_n be the our wealth at time n.

(1) Need $X_\sigma \geqslant G_\sigma$ for all finite stopping times σ. (Or equivalently $X_n \geqslant G_n$ for all n.)
(2) For (at-least) *one* stopping time σ^*, need $X_{\sigma^*} = G_{\sigma^*}$.

The arbitrage free price of this option is X_0.

Question 6.57. *Does Strategy I replicate an American option? Say σ^* is the optimal exercise time, and we create a replicating portfolio (with wealth process X) for the option with payoff G_{σ^*} at time σ^*. Suppose an investor cashes out the American option at time τ. Can we pay him?*

Question 6.58. *Does Strategy II yield the same price as Strategy I? I.e. must $X_0 = \max\{V_0^\sigma \mid \sigma \text{ is a finite stopping time }\}$?*

Question 6.59. *Is the wealth of the replicating portfolio (for an American option) uniquely determined?*

Question 6.60. *How do you find the minimal optimal exercise time, and the arbitrage free price? Let's take a simple example first.*

Theorem 6.61. *Consider the binomial model with $0 < d < 1 + r < u$, and an American option with intrinsic value G. Define*

$$V_N = G_N \,, \qquad V_n = \max\Big\{ \frac{1}{D_n} \tilde{\boldsymbol{E}}_n (D_{n+1} V_{n+1}), G_n \Big\} \,, \qquad \sigma^* = \min\{ n \leqslant N \mid V_n = G_n \} \,.$$

Then V_n is the arbitrage free price, and σ^* is the minimal optimal exercise time. Moreover, this option can be replicated.

Remark 6.62. The above is true in any complete, arbitrage free market.

Remark 6.63. In the Binomial model the above simplifies to:

$$V_n(\omega) = \max\Big\{ \frac{1}{1+r} \Big(\tilde{p} V_{n+1}(\omega', 1) + \tilde{q} V_{n+1}(\omega', -1) \Big), G_n(\omega) \Big\} \,, \qquad \text{where } \omega = (\omega', \omega_{n+1}, \omega'') \,, \quad \omega' = (\omega_1, \dots, \omega_n) \,.$$

Remark 6.64. We will prove Theorem 6.61 in the next section after proving the Doob decomposition.

Theorem 6.65. *Consider the Binomial model with $0 < d < 1 + r < u$, and a state process $Y = (Y^1, \dots, Y^d)$ such that $Y_{n+1}(\omega) = h_{n+1}(Y_n(\omega'), \omega_{n+1})$, where $\omega' = (\omega_1, \dots, \omega_n)$, $\omega = (\omega', \omega_{n+1}, \dots, \omega_N)$, and $h_0, h_1, \dots, h_N$ are N deterministic functions. Let $g_0, \dots, g_N$ be N deterministic functions, let $G_k = g_k(Y_k)$, and consider an American option with intrinsic value $G = (G_0, G_1, \dots, G_N)$. The* pre-exercise price *of the option at time n is $f_n(Y_n)$, where*

$$f_N(y) = g_N(y) \quad \textit{for } y \in \operatorname{Range}(Y_N) \,, \quad f_n(y) = \max\Big\{ g_n(y), \frac{1}{1+r} \Big(\tilde{p} f_{n+1}(h_{n+1}(y, +1)) + \tilde{q} f_{n+1}(h_{n+1}(y, -1)) \Big) \Big\} \,, \quad \textit{for } y \in \operatorname{Range}(Y_n) \,.$$

The minimal optimal exercise time is $\sigma^* = \min\{ n \mid f_n(Y_n) = g_n(Y_n) \}$.

Theorem 6.66. *Suppose the interest rate r is nonnegative. Let g be a convex function with $g(0) = 0$, and let $G_n = g(S_n)$. Consider an American option with intrinsic value $G_n = g(S_n)$. Then $\sigma^* = N$ is an optimal exercise time. That is, it is not advantageous to exercise this option early.*

Corollary 6.67. *The arbitrage free price of an American call and European call are the same.*

6.6. **Optimal Stopping.**

Definition 6.68. We say an adapted process M is a *super-martingale* if $\boldsymbol{E}_n M_{n+1} \leqslant M_n$.

Definition 6.69. We say an adapted process M is a *sub-martingale* if $\boldsymbol{E}_n M_{n+1} \geqslant M_n$.

Example 6.70. The discounted arbitrage free price of an American option is a *super-martingale* under the risk neutral measure.

Theorem 6.71 (Doob decomposition)**.** *Any adapted process can be uniquely expressed as the sum of a martingale and a predictable process that starts at* 0*. That is, if X is an adapted process there exists a unique pair of process M, A such that M is a martingale, A is predictable, $A_0 = 0$ and $X = M + A$.*

Proposition 6.72. *If X is a super-martingale, then there exists a unique martingale M and increasing predictable process A such that $X = M - A$.*

Proposition 6.73. *If X is a sub-martingale, then there exists a unique martingale M and increasing predictable process A such that $X = M + A$.*

Corollary 6.74. *If X is a super-martingale and τ is a bounded stopping time, then $\boldsymbol{E}_n X_\tau \leqslant X_{\tau \wedge n}$.*

Corollary 6.75. *If X is a sub-martingale and τ is a bounded stopping time, then $\boldsymbol{E}_n X_\tau \geqslant X_{\tau \wedge n}$.*

Theorem 6.76 (Snell)**.** *Let G be an adapted process, and define V by*

$$V_N = G_N \qquad V_n = \max\{\boldsymbol{E}_n V_{n+1}, G_n\}\,.$$

Then V is the smallest super-martingale for which $V_n \geqslant G_n$.

Proposition 6.77. *If W is any martingale for which $W_n \geqslant G_n$, and for one stopping time τ^* we have $\boldsymbol{E} W_{\tau^*} = \boldsymbol{E} G_{\tau^*}$, then we must have $W_{\tau^* \wedge n} = V_{\tau^* \wedge n}$, and $V_{\tau^* \wedge n}$ is a martingale.*

Theorem 6.78. *Let $\sigma^* = \min\{n \mid V_n = G_n\}$. Then σ^* is the minimal solution to the* optimal stopping problem *for G. Namely, $\boldsymbol{E}G_{\sigma^*} = \max_\sigma \boldsymbol{E}G_\sigma$ where the maximum is taken over all finite stopping times σ. Moreover, if $\boldsymbol{E}G_{\tau^*} = \max_\sigma \boldsymbol{E}G_\sigma$ for any other finite stopping time τ^*, we must have $\tau^* \geqslant \sigma^*$.*

Remark 6.79. By construction $V_{\sigma^* \wedge n}$ is a martingale.

Theorem 6.80. *For any $k \in \{0, \dots, N\}$, let $\sigma_k^* = \min\{n \geqslant k \mid V_n = G_n\}$. Then $\boldsymbol{E}_k G_{\sigma_k^*} = \max_{\sigma_k} \boldsymbol{E}_k G_{\sigma_k}$, where the maximum is taken over all finite stopping times σ_k for which $\sigma_k \geqslant k$ almost surely.*

Theorem 6.81. *Let $V = M - A$ be the Doob decomposition for V, and define $\tau^* = \max\{n \mid A_n = 0\}$. Then τ^* is a stopping time and is the* largest *solution to the optimal stopping problem for G.*

6.7. **American options (with proofs).** Consider the N period binomial model with $0 < d < 1 + r < u$.

Proposition 6.82. *Any American option can be replicated. That is, consider an American option with intrinsic value G. There exists a self financing portfolio X such that:*

(1) *$X_n \geqslant G_n$ for all n*
(2) *For some stopping time σ^*, we have $X_{\sigma^*} = G_{\sigma^*}$.*

Proposition 6.83. *If X is the wealth of a replicating portfolio with $X_{\sigma^*} = G_{\sigma^*}$. Then σ^* is an optimal exercise policy. Moreover, if τ^* is any optimal exercise policy, then $X_{\tau^*} = G_{\tau^*}$*

Corollary 6.84 (Uniqueness)**.** *If X, and Y are wealth of two replicating portfolios for an American option with intrinsic value G, then for any optimal exercise time σ^* we must have $\mathbf{1}_{n \leqslant \sigma^*} X_n = \mathbf{1}_{n \leqslant \sigma^*} Y_n$.*

Proposition 6.85. *Let $V_N = G_N$, and $V_n = \max\{G_n, D_n^{-1} \tilde{E}_n V_{n+1}\}$. Then V_n is the arbitrage free price of the American option. That is, the market remains arbitrage free if we are allowed to trade an American option at price V_n.*

7. Fundamental theorems of Asset Pricing

7.1. Markets with multiple risky assets.

(1) $\Omega = \{1, \dots, M\}^N$ is a probability space representing N rolls of M-sided dies, and p is a probability mass function on Ω.
(2) The die rolls need not be i.i.d.
(3) Consider a financial market with $d+1$ assets $S^0, S^1, \dots, S^d$. (S^k_n denotes the price of the k-th asset at time n.)
(4) For $i \in \{1, \dots, d\}$, S^i is an adapted process (i.e. S^i_n is $\mathcal{F}_n$-measurable).
(5) The 0-th asset S^0 is assumed to be a *risk free* bank/money market:
    (a) Let r_n be an adapted process specifying the interest rate at time n.
    (b) Let $S^0_0 = 1$, and $S^0_{n+1} = (1 + r_n) S^0_n$. (Note S^0 is predictable.)
    (c) Let $D_n = (S^0_n)^{-1}$ be the discount factor (D_n dollars at time 0 becomes 1 dollar at time n).
(6) Let $\Delta_n = (\Delta^0_n, \dots, \Delta^d_n)$ be the position at time n of an investor in each of the assets $(S^0_n, \dots, S^d_n)$.
(7) The wealth of an investor holding these assets is given by $X_n = \Delta_n \cdot S_n \stackrel{\text{def}}{=} \sum_{i=0}^d \Delta^i_n S^i_n$.

Definition 7.1. Consider a portfolio whose positions in the assets at time n is Δ_n. We say this portfolio is *self-financing* if Δ_n is adapted, and $\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}$.

7.2. First fundamental theorem of asset pricing.

Definition 7.2. We say the market is arbitrage free if for any self financing portfolio with wealth process X, we have: $X_0 = 0$ and $X_N \geqslant 0$ implies $X_N = 0$ almost surely.

Definition 7.3. We say $\tilde{\boldsymbol{P}}$ is a *risk neutral measure* if $\tilde{\boldsymbol{P}}$ is equivalent to $\boldsymbol{P}$ and $\tilde{\boldsymbol{E}}_n(D_{n+1} S^i_{n+1}) = D_n S^i_n$ for every $i \in \{0, \dots, d\}$.

Theorem 7.4. *The market defined in Section 7.1 is arbitrage free if and only if there exists a risk neutral measure.*

Lemma 7.5. *If $\tilde{\boldsymbol{P}}$ is a risk neutral measure, then the discounted wealth of any self financing portfolio is a $\tilde{\boldsymbol{P}}$-martingale.*

Proof of Theorem 7.4 (Existence of a risk neutral measure implies no-arbitrage).

Corollary 7.6. *Suppose the market has a risk neutral measure $\tilde{\boldsymbol{P}}$. Let V_N be a $\mathcal{F}_N$-measurable random variable and consider an security that pays V_N at time N. Then $V_n = D_n^{-1}\tilde{\boldsymbol{E}}_n(D_N V_N)$ is a arbitrage free price at time $n \leqslant N$. (i.e. allowing you to trade this security in the market with price V_n at time n keeps the market arbitrage free).*

Remark 7.7. We do not, however, know that the security can be replicated.

Lemma 7.8. *Suppose the market has no arbitrage, and X is the wealth process of a self-financing portfolio. If for any n, $X_n = 0$ and $X_{n+1} \geqslant 0$, then we must have $X_{n+1} = 0$ almost surely.*

Lemma 7.9. *Suppose we find an equivalent measure $\tilde{\boldsymbol{P}}$ such that whenever $\Delta_n \cdot S_n = 0$, we have $\tilde{\boldsymbol{E}}_n(\Delta_n \cdot S_{n+1}) = 0$, then $\tilde{\boldsymbol{P}}$ is a risk neutral measure.*

Lemma 7.10. *Suppose $\tilde{p}$ is a probability mass function such that $\tilde{p}(\omega) = \tilde{p}_1(\omega_1)\tilde{p}_2(\omega_1,\omega_2)\cdots\tilde{p}_N(\omega_1,\ldots,\omega_N)$. If X_{n+1} is $\mathcal{F}_{n+1}$-measurable, then*

$$\tilde{\boldsymbol{E}}_n X_{n+1}(\omega) = \sum_{i=1}^{M} \tilde{p}_{n+1}(\omega', j) X_{n+1}(\omega', j)\,, \qquad \textit{where} \qquad \omega' = (\omega_1, \ldots, \omega_n)\,, \omega = (\omega', \omega_{n+1}, \omega_{n+1}, \ldots, \omega_N)$$

Lemma 7.11. *Define $\bar{Q} \stackrel{\text{def}}{=} \{v \in \mathbb{R}^M \mid v_i \geqslant 0 \ \forall i \in \{1, \ldots, M\}\}$, and $\mathring{Q} \stackrel{\text{def}}{=} \{v \in \mathbb{R}^M \mid v_i > 0 \ \forall i \in \{1, \ldots, M\}\}$. Let $V \subseteq R^M$ be a subspace.*

1. *$V \cap \bar{Q} = \{0\}$ if and only if there exists $\hat{n} \in \mathring{Q}$ such that $|\hat{n}| = 1$ and $\hat{n} \perp V$.*
2. *The unit normal vector $\hat{n} \in \mathring{Q}$ is unique if and only if $V \cap \bar{Q} = \{0\}$ and $\dim(V) = M - 1$.*

Remark 7.12. This can be proved using the *Hyperplane separation theorem* used in convex analysis.

Proof of Theorem 7.4 (No arbitrage implies existence of a risk neutral measure).

7.3. **Second fundamental theorem.**

Definition 7.13. A market is said to be *complete* if every derivative security can be hedged.

Theorem 7.14. *The market defined in Section 7.1 is complete and arbitrage free if and only if there exists a* unique *risk neutral measure.*

Lemma 7.15. *The market is complete if and only if for every $\mathcal{F}_{n+1}$-measurable random variable X_{n+1}, there exists a (not necessarily unique) $\mathcal{F}_n$ measurable random vector $\Delta_n = (\Delta_n^0, \dots, \Delta_n^d)$ such that $X_{n+1} = \Delta_n \cdot S_{n+1}$.*

Proof of Theorem 7.14

7.4. **Examples and Consequences.**

Proposition 7.16. *Suppose the market model Section 7.1 is complete and arbitrage free, and let $\tilde{\boldsymbol{P}}$ be the unique risk neutral measure. If $D_n X_n$ is a $\tilde{\boldsymbol{P}}$ martingale, then X_n must be the wealth of a self financing portfolio.*

Remark 7.17. We've already seen in Lemma 7.5 that if a (not necessarily unique) risk neutral measure exists, then the discounted wealth of any self financing portfolio must be a martingale under it.

Remark 7.18. All pricing results/formulae we derived for the Binomial model that only relied on the analog of Proposition 7.16 will hold in complete arbitrage free markets.

Question 7.19. *Consider a market consisting of a bank with interest rate r, and two stocks with price processes S^1, S^2. At each time step we flip two independent coins. The price of the i-th stock ($i \in \{1, 2\}$) changes by factor u_i, or d_i depending on whether the i-th coin is heads or tails. When is this market arbitrage free? When is this market complete?*

Question 7.20. *Consider now repeated rolls of a 3-sided die and for $i \in \{1, 2\}$, suppose $S^i_{n+1} = f_{i,j} S^i_n$, if $\omega_{n+1} = j$. How do you find the risk neutral measure? Find conditions when this market is complete and arbitrage free.*

8. Black-Scholes Formula

(1) Suppose now we can trade *continuously in time.*
(2) Consider a market with a bank and a stock, whose spot price at time t is denoted by S_t.
(3) The *continuously compounded interest rate* is r (i.e. money in the bank grows like $\partial_t C(t) = rC(t)$).
(4) Assume liquidity, neglect transaction costs (frictionless), and the borrowing/lending rates are the same.
(5) In the *Black-Scholes* setting, we model the stock prices by a *Geometric Brownian motion* with parameters α (the mean return rate) and σ (the volatility).
(6) The price at time t of a European call with maturity T and strike K is given by

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x))\,,$$

$$\text{where}\quad d_\pm = \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \qquad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy\,.$$

(7) Can be obtained as the limit of the Binomial model as $N \to \infty$ by choosing:

$$r_{\text{binom}} = \frac{r}{N}\,, \qquad u = u_N = 1 + \frac{r}{N} + \frac{\sigma}{\sqrt{N}} \qquad d = d_N = 1 + \frac{r}{N} - \frac{\sigma}{\sqrt{N}}$$

9. Recurrence of Random Walks

- Let ξ_n be a sequence of i.i.d. coin flips with $\boldsymbol{P}(\xi_n = 1) = \boldsymbol{P}(\xi_n = -1) = 1/2$.
- Simple random walk: $S_n = \sum_1^n \xi_k$ (i.e. $S_0 = 0$, $S_{n+1} = S_n + \xi_{n+1}$).

Definition 9.1. The process S_n is recurrent at 0 if $\boldsymbol{P}(S_n = 0 \text{ infinitely often })$.

Question 9.2. *Is the random walk (in one dimension) recurrent at* 0*? How about at any other value?*

Question 9.3. *Say ξ_n are i.i.d. random vectors in $\mathbb{R}^d$ with $\boldsymbol{P}(\xi_n = \pm e_i) = \frac{1}{2d}$. Set $S_n = \sum_1^n \xi_k$. Is S_n recurrent at 0?*

Theorem 9.4. *The simple random walk in $\mathbb{R}^d$ is recurrent for $d = 1, 2$ and transient for $d \geqslant 3$.*

- Let $\tau_1 = \min\{n > 0 \mid S_n = 0\}$, be the first time S returns to 0.
- Let $\tau_2 = \min\{n > \tau_1 \mid S_n = 0\}$, be the first time after τ_1 that S returns to 0.
- Let $\tau_{k+1} = \min\{n > \tau_k \mid S_n = 0\}$, be the first time after τ_k that S returns to 0.

Lemma 9.5. *S is recurrent at 0 if and only if $\boldsymbol{P}(\tau_0 < \infty) = 1$.*

Lemma 9.6. *$\boldsymbol{P}(\tau_0 < \infty) = 1$ if and only if $\sum \boldsymbol{P}(S_n = 0) = \infty$.*

Proof.

Theorem 9.7. *$\boldsymbol{P}(S_{2m} = 0) = O(1/m^{d/2})$. Consequently, the random walk is* recurrent *for $d \leqslant 2$, and* transient *for $d \geqslant 3$.*

Lemma 9.8 (Sterling's formula)**.** *For large n, we have*

$$n! \approx \sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n = \sqrt{2\pi}\exp\Big(n\ln n - n + \frac{\ln n}{2}\Big).$$

Proof of Theorem 9.7 for $d = 1$:

Remark 9.9. Recall the Gambler's ruin example (Question 6.50): Let ξ_n be i.i.d. random variables with mean 0, and let $X_n = \sum_1^n \xi_k$. Let $\tau = \min\{n \mid X_n = 1\}$. Theorem 9.7 proves $\tau < \infty$ almost surely. We proved earlier $\boldsymbol{E}X_\tau = 1$ and $\lim_{N\to\infty} \boldsymbol{E}X_{\tau\wedge N} = 0$.

Theorem 9.10. *Consider the Gamblers ruin example, with $\tau = \min\{n \mid X_n = 1\}$. Then*

$$\boldsymbol{E}\tau = \infty \qquad \text{and} \qquad \boldsymbol{P}(\tau = 2n-1) = (-1)^{n-1}\binom{1/2}{n} \approx \frac{C}{n^{3/2}}$$

Remark 9.11. Let $M_n = \min\{X_{\tau \wedge k} \mid k \leqslant n\}$. Then $\boldsymbol{E} M_\tau = -\infty$. Thus, this strategy will take (on average) an infinite time before you win \$1. During that time your expected maximum loss is $-\infty$.

Lemma 9.12. *Let* $F(x) = \boldsymbol{E} x^\tau$. *Then* $F(x) = \frac{1}{x}(1 - \sqrt{1 - x^2})$.

Proof of Theorem 9.10