LECTURE NOTES ON DISCRETE TIME FINANCE
FALL 2020

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Contents

Note: The page numbers and links will not be correct in the annotated version.

1. Preface.

These are the slides I used while teaching this course in 2020. I projected them (spaced out) in class, and filled in the proofs by writing over them with a tablet. Both the annotated version of these slides with handwritten proofs, and the compactified un-annotated version can be found on the class website. The LaTeXsource of these slides is also available on git.

21 - 370

2. Syllabus Overview

- Class website and full syllabus: `http://www.math.cmu.edu/~gautam/sj/teaching/2020-21/370-dtime-finance`
- TA's: ~~Jonghwa Park~~ <jonghwap@andrew.cmu.edu>, Karl Xiao <kzx@andrew.cmu.edu>, Hongyi Zhou <hongyizh@andrew.cmu.edu>
- Homework Due: Every Wednesday, before class (on Gradescope)
- Midterms: Wed Sep 29, 5th week, and Wed Nov 3rd, 10th week (self proctored, can be taken any time)
- **Zoom lectures:**
  - ▷ Please enable video. (It helps me pace lectures).
  - ▷ Mute your mic when you're not speaking. Use headphones if possible. Consent to be recorded.
  - ▷ If I get disconnected, check your email for instructions.
- **Homework:**
  - ▷ Good quality scans please! Use a scanning app, and not simply take photos. (I use Adobe Scan.)
  - ▷ 20% penalty if turned in within an hour of the deadline. 100% penalty after that.
  - ▷ Two homework assignments can be turned in 24h late without penalty.
  - ▷ Bottom 2 homework scores are dropped from your grade (personal emergencies, other deadlines, etc.).
  - ▷ Collaboration is encouraged. Homework is not a test – ensure you learn from doing the homework.
  - ▷ You must write solutions independently, and can only turn in solutions you fully understand.
- **Exams:**
  - ▷ Can be taken at any time on the exam day. Open book. Use of internet allowed.
  - ▷ Collaboration is forbidden. You may not seek or receive assistance from other people. (Can search forums; but may not post.)
  - ▷ Self proctored: Zoom call. Record yourself, and your screen to the cloud.
  - ▷ Share the recording link; also download a copy and upload it to the designated location immediately after turning in your exam.

- **Academic Integrity**
  - ▷ Zero tolerance for violations (automatic **R**).
  - ▷ Violations include:
    - – Not writing up solutions independently and/or plagiarizing solutions
    - – Turning in solutions you do not understand.
    - – Seeking, receiving or providing assistance during an exam.
    - – Discussing the exam on the exam day (24h). Even if you have finished the exam, others may be taking it.
  - ▷ All violations will be reported to the university, and they may impose additional penalties.
- **Grading:** 30% homework, 20% each midterm, 30% final.

3. Replication, and Arbitrage Free Pricing

- Start with a *financial market* consisting of traded assets (stocks, bonds, money market, options, etc.)
- We model the price of these assets through random variables (stochastic processes).
- **No Arbitrage Assumption:**
  - ▷ In order to make money, you have to take risk. (Can't make something out of nothing.)
  - ▷ There doesn't exist a trading strategy with $X_0 = 0$, $X_n \geqslant 0$ and $\boldsymbol{P}(X_n > 0) > 0$.
- Now consider a non-traded asset Y (e.g. an option). How do you price it?
- *Arbitrage free price:* V_0 is the arbitrage free price of Y, if given the opportunity to trade Y at price V_0, the market remains arbitrage free.

X ← stock

M ← money market

Market

← arb free

Y → new asset Y.

V_0 = $\underbrace{\text{arb free price of } Y}$ if

New market

want arb free

X

M

Y ← introduce at price V_0.

- How do you compute the arbitrage free price? Replication:
  - ▷ Say the non-traded asset pays V_N at time N (e.g. call options).
  - ▷ Say you can *replicate* the payoff through a trading strategy $X_0, \dots, \boxed{X_N = V_N}$ (using only traded assets).
  - ▷ $\boxed{\text{Then the arbitrage free price is uniquely determined, and must be } \boxed{X_0}.}$

Question 3.1. *Is the arbitrage free price always unique?*

Eg: $S_n \longrightarrow$ price at time n of a stock.

Bank $\longrightarrow$ interest rate r.

Market {Stock + Bank}.

NTA: Call option pays $(S_N - K)^+$ at time N.

strike price

Maturity

Say through some seq of trade we have a strategy wealth $X_0, X_1, \dots X_N = (S_N - K)^+$ (Replicating strat)

Why is AFP of the non tradable asset X_0 (initial wealth of the Replicating strategy).

Reason: V_0 = AFP of the

X_0 = init wealth of Rep strategy.

Say $X_0 < V_0$. Then Borrow X_0 \$ & buy the Hedging strat.
copy

and Short the new asset.

At time 0, have $(V_0 - X_0)$ \$. $\longrightarrow$ Put in bank. "

At time $N \longrightarrow$ wealth $= \begin{cases} X_N & \text{From trading strat} \\ -V_N & \longleftarrow \text{shorting NTA} \\ (V_0 - X_0)(1+r)^N & \longleftarrow \text{Bank} \end{cases}$

$X_N = V_N$

Wealth at time $N = \underbrace{(V_0 - X_0)(1+r)^N > 0}$.

arbitrage

Theorem 3.2. *The arbitrage free price is* unique *if and only if there is a replicating strategy! In this case, the arbitrage free price is exactly the initial capital of the replicating strategy.*

Proof. We already proved that if a replicating strategy exists then the arbitrage free price is unique. The other direction is harder, and will be done later. □

More on this later.

Question 3.3. *If a replicating strategy exists, must it be unique?*

NO! But, the initial wealth of the replicating strategy has to be unique (it is the AFP!)

① HW 1 is online.

$\boxed{\text{② Hint on Q2(f) is on the discussion board}}$

<u>Last time:</u> No arb: To make \$ you need to take risk.

$\iff$ If $X_0 = 0$, know $X_n \geq 0$ then must have $X_n = 0$ (almost surely).

<u>AFP:</u>

$S^{(1)}$

$\vdots$

$S^{(m)}$

Market (arb free)

V ← non traded asset

AFP: Given the opportunity to trade the new asset at price V_0, the market remains arb free!

Question 3.4. *Consider a financial market with a money market account with interest rate r, and a stock. Let $K > 0$. A* forward contract *requires the holder to buy the stock at price K at maturity time N. What is the arbitrage free price at time 0?*

Payoff : Let S_n = stock price at time n.

at maturity forward contract pays $S_N - K$

To compute AFP → Replicate it.

Use only tradable assets, Start with X_0 \$. & end with $(S_N - K)$ \$.

↓

AFP.

Strategy : ① Buy the stock (costs S_0 \$) (worth S_N \$ at time N)

(2) Put $\frac{K}{(1+r)^N}$ \$ in the bank.

(worth K \$ at time N)

(3) Don't trade until maturity.

$$X_0 = \text{wealth at time } 0 = \left[S_0 - \frac{K}{(1+r)^N} \right].$$

$$X_N = \text{wealth at time } N = S_N - K = \text{payoff of F.C.}$$

$\boxed{\Rightarrow \text{ AFP of the forward contract at time 0 is } S_0 - \frac{K}{(1+r)^N}}$

4. Binomial model (one period)

Say we have access to a money market account with interest rate r. The *binomial model* dictates that the stock price varies as follows. Let $p \in (0,1)$, $q = 1 - p$, $0 < d < u$ (up and down factors). Flip a coin that lands heads with probability p, and tails with probability q. When the coin lands heads, the stock price changes by the factor u, and when it lands tails it changes by the factor $\cancel{p}$ d.

Question 4.1. *When is there arbitrage in this market?*

$$\boxed{\text{No arb} \iff d < 1 + r < u}$$

If $d > 1+r$ then
buy 1 share of stock
Put $-S_0$ \$ in bank.
⇒ Arbitrage

If instead $1+r > u$
Short stock & put — S_0 \$ in bank
⇒ Arbitrage.

Also need to check there is no arb if $\boxed{d < 1+r < u.}$

Start with $X_0 = 0$ $\begin{cases} \Delta_0 & \text{shares of stock} \\ -\Delta_0 S_0 & \text{in bank.} \end{cases}$

Wealth at time $0 = \Delta_0 S_0 + (-\Delta_0 S_0) = 0.$

$$\text{Wealth at time } 1 = \Delta_0 S_1 - \Delta_0 S_0 (1+r)$$

$$= \Delta_0 \left(S_1 - (1+r) S_0 \right) = \begin{cases} \Delta_0 \left(u - (1+r)\right) S_0 & \text{if heads} \\ \Delta_0 \left(d - (1+r)\right) S_0 & \text{if tails.} \end{cases}$$

Want $X_1 \geq 0$. Note $d < 1+r < u$

$\Rightarrow u-(1+r) > 0$ & $d-(1+r) < 0$.

$X_1 \geq 0$ if heads can only happen if $\Delta_0 \geq 0$

$X_1 \geq 0$ if tails " " " " $\Delta_0 \leq 0$

$X_1 \geq 0$ regardless of coin flip can only happen if $\Delta_0 = 0$.

$\therefore$ If $X_0 = 0$ & $X_1 \geq 0$ $\Rightarrow X_1 = 0$ (No arb).

$(d < 1+r < u)$

Question 4.2. *If a security pays V_1 at time 1, what is the arbitrage free price at time 0. (V_1 can depend on whether the coin flip is heads or tails).*

Find AFP by replication.

Start with X_0 \$. $\to$ Δ_0 shares of stock (costs $\Delta_0 S_0$) at time 0.

$\to$ Rest cash. $(X_0 - \Delta_0 S_0)$.

X_1 = wealth at time 1 $= \Delta_0 S_1 + (X_0 - \Delta_0 S_0)(1+r) \overset{\text{Want}}{=} V_1.$

$$X_1 = \Delta_0 \big(S_1 - (1+r) S_0\big) + X_0 (1+r) \overset{\text{Want}}{=} V_1.$$

$S_1(H)$ = stock price at time 1 if heads $= u S_0$

$S_1(T)$ = " " " " " " tails $= d S_0$.

$V_1(H)$ = sec price at time 1 if heads

$V_1(T)$ = " " " " " tails.

$$X_1(H) = \Delta_0 \big(u - (1+r) \big) S_0 + X_0 (1+r) \overset{\text{Want}}{=} V_1(H).$$

$$X_1(T) = \Delta_0 \big(d - (1+r) \big) S_0 + X_0 (1+r) \overset{\text{Want}}{=} V_1(T).$$

a

$\boxed{\text{Is this possible?}}$ Can choose Δ_0 & X_0

↳ Yes! (2 eq 2 unknowns) → Δ_0 & X_0

Will solve next time & find Δ_0, X_0.

ENABLE VIDEO IF YOU CAN

(& even if you can't !! ☺)

4. Binomial model (one period)

Say we have access to a money market account with interest rate r. The *binomial model* dictates that the stock price varies as follows. Let $p \in (0,1)$, $q = 1 - p$, $0 < d < u$ (up and down factors). Flip a coin that lands heads with probability p, and tails with probability q. When the coin lands heads, the stock price changes by the factor u, and when it lands tails it changes by the factor d.

Question 4.1. *When is there arbitrage in this market?*

Coin flip

uS_0

Heads (p)

S_0

(stock price at time 0)

Tails ($q = 1-p$)

dS_0

Stock

Money market

Interest rate r

Market

No Arb $\iff d < 1+r < u$

Question 4.2. *If a security pays V_1 at time 1, what is the arbitrage free price at time 0. (V_1 can depend on whether the coin flip is heads or tails).*

Say Binom model NO arb ($d < 1+r < u$).

Replicate V_1. Start with X_0 wealth

$\to \Delta_0$ shares of Stock.

$\to X_0 - \Delta_0 S_0$ cash.

Goal: Find X_0 & Δ_0 $\ni$ wealth at time 1 $= V_1$ (payoff of sec).

If we do this then X_0 = AFP.

(1) X_1 = wealth at time 1 $= \Delta_0 S_1 + (X_0 - \Delta_0 S_0)(1+r)$

$$= \Delta_0\left(S_1 - (1+r)S_0\right) + (1+r)X_0 \overset{\text{Want}}{=} V_1 .$$

$\Leftrightarrow$ (a) If heads: $\Delta_0\left(uS_0 - (1+r)S_0\right) + (1+r)X_0 = V_1(H)$.

(b) If tails: $\Delta_0\left(dS_0 - (1+r)S_0\right) + (1+r)X_0 = V_1(T)$.

2 Eq. 2 Unknowns (X_0 & Δ_0). Solve

(2) To solve: find $\tilde{p}$ & $\tilde{q}$ + $\boxed{\tilde{p} + \tilde{q} = 1}$

& $\tilde{p}\,\underbrace{S_1(H)}_{uS_0} + \tilde{q}\,\underbrace{S_1(T)}_{dS_0} = (1+r)S_0$

$$\boxed{\Leftrightarrow \tilde{p}u + \tilde{q}d = 1+r}$$

③ $\tilde{p}ⓐ + \tilde{q}ⓑ \Rightarrow (1+r)X_0 = \tilde{p}V_1(H) + \tilde{q}V_1(T)$

$$\boxed{\Rightarrow X_0 = \frac{\tilde{p}V_1(H) + \tilde{q}V_1(T)}{1+r}}$$

4. Find Δ_0: Take ⓐ − ⓑ.

$$\Rightarrow \Delta_0 (u-d) S_0 = V_1(H) - V_1(T)$$

$$\boxed{\Leftrightarrow \Delta_0 = \frac{V_1(H) - V_1(T)}{(u-d) S_0}}$$

5. $\tilde{p} u + \tilde{q} d = 1+r \Leftrightarrow \tilde{p} u + (1-\tilde{p}) d = 1+r$

$$\Leftrightarrow \tilde{p}(u-d) = 1+r-d \Leftrightarrow \boxed{\tilde{p} = \frac{1+r-d}{u-d}}$$

$$\tilde{q} = 1 - \tilde{p} = \frac{u - (1+r)}{u - d}.$$

Rem: $\tilde{p}$ & $\tilde{q}$ called the "Risk neutral Probabilities".

(1) Expected return of Stock after time 1.

$$= p\, S_1(H) + q\, S_1(T) = \big(\underbrace{pu + qd}\big)\, S_0$$

(2) Suppose now the coin flips heads with prob $\tilde{p}$
& tails with prob $\tilde{q}$.

Expected return of stock at time $1 = \tilde{p} S_1(H) + \tilde{q} S_1(T)$

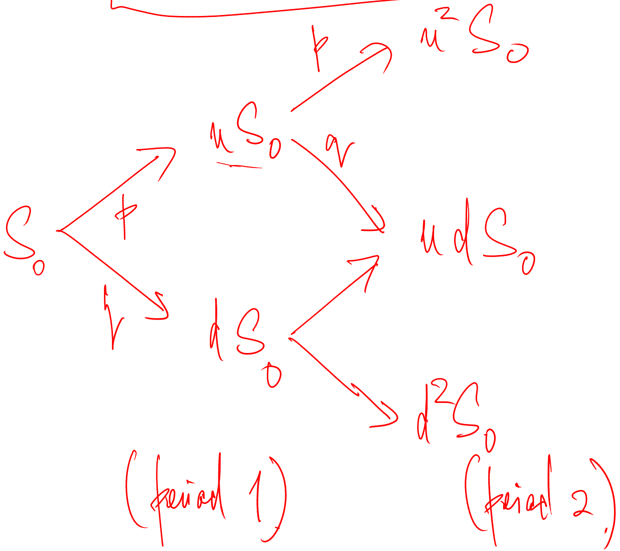
$$= (1+r) S_0$$

= Same return as putting money in bank.

Note AFP $= X_0 = \dfrac{\tilde{p} V_1(H) + \tilde{q} V_1(T)}{\underline{1+r}} = \dfrac{1}{1+r}$ (Expected return of security, if you flip heads with prob $\tilde{p}$ & tails " " $\tilde{q}$).

Expected return under the risk neutral measure.

Question 4.3. *What's an N period version of this model? Do we have the same formulae?*



$$S_n = S_0 \underbrace{\left(u^{\#\text{ heads}}\right)\left(d^{\#\text{ tails}}\right)}_{n\text{ - iid coin flips.}}.$$

Goal: Analyse the n period case thoroughly.

1. Securities that don't expire at a fixed time.
2. American options.

5. A quick introduction to probability

(325 → Conseq. → Either you know prob or you're in 325).

Let $N \in \mathbb{N}$ be large (typically the maturity time of financial securities).

Definition 5.1. The *sample space* is the set $\Omega = \{(\omega_1, \dots, \omega_N) \mid \text{each } \omega_i \text{ represents the outcome of a coin toss (or die roll).}\}$

- E.g. $\omega_i \in \{H, T\}$, or $\omega_i \in \{\pm 1\}$.
- Coins / dice don't have to be identical: Pick $M_1, M_2, \dots, \in \mathbb{N}$, and can require $\omega_i \in \{1, \dots, M_i\}$.
- Usually in probability the *sample space* is simply a set; however, for our purposes it is more convenient to consider "coin toss spaces" as we defined above.

Definition 5.2. A *sample point* is a point $\omega = (\omega_1, \dots, \omega_N) \in \Omega$.

→ $\omega \in \Omega$ is a sample point & write $\omega = (\omega_1, \omega_2, \dots \omega_N)$

Definition 5.3. A $\boxed{\textit{probability mass function}}$ is a function $p \colon \Omega \to [0, 1]$ such that $\boxed{\sum_{\omega \in \Omega} p(\omega) = 1.}$

Definition 5.4. An event is a subset of Ω. Define $\boldsymbol{P}(A) = \sum_{\omega \in A} p(\omega)$.

($p(\omega)$ = prob of $\{\omega\}$ occurring)

$A \subset \Omega$ some event

$P(A)$ = prob A occurs $= \sum_{\omega \in A} p(\omega)$

Please enable videos if possible!

5. A quick introduction to probability

This is just a quick reminder, and specific to our situation (coin toss spaces). You should have already taken a probability course, or be co-enrolled in one. The only thing we will cover in any detail is conditional expectation.

Let $N \in \mathbb{N}$ be large (typically the maturity time of financial securities).

Definition 5.1. The *sample space* is the set $\Omega = \{(\omega_1, \dots, \omega_N) \mid \text{each } \omega_i \text{ represents the outcome of a coin toss (or die roll).}\}$

- ▷ E.g. $\omega_i \in \{H, T\}$, or $\omega_i \in \{\pm 1\}$.
- ▷ Coins / dice don't have to be identical: Pick $M_1, M_2, \dots, \in \mathbb{N}$, and can require $\omega_i \in \{1, \dots, M_i\}$.
- ▷ Usually in probability the *sample space* is simply a set; however, for our purposes it is more convenient to consider "coin toss spaces" as we defined above.

Definition 5.2. A *sample point* is a point $\omega = (\omega_1, \dots, \omega_N) \in \Omega \in \Omega$.

Definition 5.3. A *probability mass function* is a function $p \colon \Omega \to [0, 1]$ such that $\sum_{\omega \in \Omega} p(\omega) = 1$.

Definition 5.4. An event is a subset of Ω. Define $\boldsymbol{P}(A) = \sum_{\omega \in A} p(\omega)$.

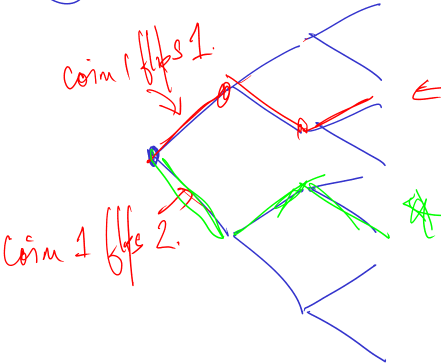
Visualize Ω for coin tosses: $\{\pm 1\}$ or $\{H, T\}$.

N iid coins $\quad \omega_i \in \{1, 2\} \quad \forall i \in \{1, \dots N\}$

independent, identically distributed.

Say $N = 3$. $\quad \omega \in \Omega$. $\quad \omega = (1, 2, 1)$

$\omega = (2, 1, 2)$



5.1. Random Variables and Independence.

Definition 5.5. A *random variable* is a function $X : \underline{\Omega} \to \underline{\mathbb{R}}$.

Question 5.6. *What is the random variable corresponding to the outcome of the n^{th} coin toss?*

$$\Omega = \left\{ \omega = (\omega_1, \dots \omega_{\underline{N}}) \;\middle|\; \omega_i \in \{1, 2\} \right\}$$

2 sided coins.

$X_{\underline{n}}(\omega)$ = RV corresponding to the n^{th} coin toss.

$= \omega_n$ (where $\omega = (\omega_1, \omega_2, \dots \omega_N)$),

Definition 5.7. The *expectation* of a random variable X is $\boldsymbol{E}X = \sum X(\omega)p(\omega)$.

Remark 5.8. Note if $\mathrm{Range}(X) = \{x_1, \ldots, x_n\}$, then $\boldsymbol{E}X = \sum X(\omega)p(\omega) = \boxed{\sum_1^n x_i \boldsymbol{P}(X = x_i)}$.

Definition 5.9. The *variance* of a random variable is $\boxed{\mathrm{Var}(X) = \boldsymbol{E}(X - \boldsymbol{E}X)^2}$.

Remark 5.10. Note $\mathrm{Var}(X) = \boxed{\boldsymbol{E}X^2 - (\boldsymbol{E}X)^2}$.

$$\{X = x_i\} = \{\omega \in \Omega \mid X(\omega) = x_i\}$$

Notation convention EX^2 ALWAYS mean $E(X^2)$

and NOT $(EX)^2$.

Definition 5.11. Two events are independent if $\boxed{\boldsymbol{P}(A \cap B) = \boldsymbol{P}(A)\boldsymbol{P}(B).}$ $\Leftarrow$ $P(A|B) = P(A)$

Definition 5.12. The events $A_1, \ldots, A_n$ are independent if for any sub-collection $A_{i_1}, \ldots, A_{i_k}$ we have

$$\boldsymbol{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \boldsymbol{P}(A_{i_1})\boldsymbol{P}(A_{i_2}) \cdots \boldsymbol{P}(A_{i_k})\,.$$

Remark 5.13. When $n > 2$, it is not enough to only require $\boldsymbol{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \boldsymbol{P}(A_1)\boldsymbol{P}(A_2) \cdots \boldsymbol{P}(A_n)$

Eg: A_1, A_2, A_3 are ind if (1) $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$

& $P(A_1 \cap A_2) = P(A_1)P(A_2)$ & $P(A_1 \cap A_3) = P(A_1)P(A_3)$

& $P(A_2 \cap A_3) = P(A_2)P(A_3)$

Definition 5.14. Two random variables are independent if $\boldsymbol{P}(X = x,\ Y = y) = \boldsymbol{P}(X = x)\boldsymbol{P}(Y = y)$ for all $x, y \in \mathbb{R}$.

Definition 5.15. The random variables $X_1, \ldots, X_n$ are independent if for all $x_1, \ldots, x_n \in \mathbb{R}$ we have

$$\boldsymbol{P}(X_1 = x_1,\ X_2 = x_2, \ldots, X_n = x_n) = \boldsymbol{P}(X_1 = x_1)\boldsymbol{P}(X_2 = x_2) \cdots \boldsymbol{P}(X_n = x_n)\,.$$

Remark 5.16. Independent random variables are uncorrelated, but not vice versa.

Proposition 5.17. *The coin tosses in our setup are all independent, if and only if, there exists functions* $p_1, \ldots, p_N$ *such that*

$$p(\omega) = p_1(\omega_1)p_2(\omega_2) \cdots p_N(\omega_N)\,.$$

Notation convention: Capital letters $\longrightarrow$ RV's

small letters $\longrightarrow$ values they take on.

Pf: Set $X_n(\omega) = \omega_n$ (outcome of nth coin toss).

Coin tosses are indep

$\Leftrightarrow$ The R.V's $X_1, \dots X_n$ are indep.

$\Leftrightarrow$ $\forall\, \omega_1, \omega_2, \dots \omega_N \in \mathbb{R}$ we have

$$\underbrace{P(X_1 = \omega_1 \;\&\; X_2 = \omega_2 \;\dots\; X_N = \omega_N)} = P(X_1 = \omega_1)\, P(X_2 = \omega_2) \cdots P(X_N = \omega_n)$$

$$\parallel$$

$$P(\{\underbrace{(\omega_1, \omega_2, \dots \omega_N)}_{\omega}\})$$

$$\parallel$$

$$p(\omega) \qquad = \qquad \underbrace{P(X_1 = \omega_1)}_{\text{call this } p_1(\omega_1)} \cdots \underbrace{P(X_N = \omega_N)}_{\text{call this } p_N(\omega_N)}$$

QED

5.2. Filtrations.

- Let $N \in \mathbb{N}$, $d_1, \dots, d_N \in \mathbb{N}$, $\Omega = \{1, \dots, d_1\} \times \{1, \dots, d_2\} \times \cdots \times \{1, \dots, d_N\}$.
- That is $\Omega = \{\omega \mid \omega = (\omega_1, \dots, \omega_N),\ \omega_i \in \{1, \dots, d_i\}\}$.
- $d_n = 2$ for all n corresponds to flipping a two sided coin at every time step.

$\mathcal{F}_1 \subseteq \mathcal{P}(\Omega)$

Power set of Ω.

Definition 5.18. We define a *filtration* on Ω as follows:

▷ $\mathcal{F}_0 = \{\emptyset, \Omega\}$. ← Info you have before any coin flips.

▷ $\mathcal{F}_1$ = all events that can be described by only the first coin toss (die roll). E.g. $A = \{\omega \mid \omega_1 = H\} \in \mathcal{F}_1$.

▷ $\mathcal{F}_n$ = all events that can be described by only the first n coin tosses.

More precisely, given $\omega = (\omega_1, \dots, \omega_N) \in \Omega$ and $n \in \{0, \dots, N\}$ define

$$\Pi_n(\omega) = \{\omega' \in \Omega \mid \omega' = (\omega'_1, \dots, \omega'_N) \text{ and } \omega'_i = \omega_i \text{ for all } i \leqslant n\}.$$

Now $\mathcal{F}_n$ is defined by $\mathcal{F}_n \stackrel{\text{def}}{=} \left\{ A \subseteq \Omega \,\middle|\, A = \bigcup_{i=1}^{k} \Pi_n(\omega^i),\ \omega^1, \dots, \omega^k \in \Omega \right\}$

Remark 5.19. Note $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_N = \mathcal{P}(\Omega)$.

Question 5.20. *Let* $\Omega = \{H, T\}^3 \cong \{1, 2\}^3$. *What are* $\mathcal{F}_0, \dots, \mathcal{F}_3$*?*

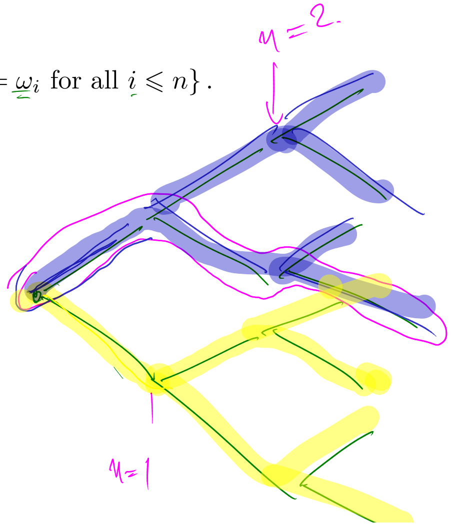
$\omega^1, \omega^2 \dots \omega^k \to k$ elements of Ω

ω

$\omega^1 = (\omega^1_1, \omega^1_2, \dots \omega^1_N)$

$n = 1$

$\Pi_1(\omega) =$



Definition 5.21. Let $n \in \{0, \dots, N\}$. We say a random variable X is $\mathcal{F}_n$-measurable if $X(\omega)$ only depends on $\omega_1, \dots, \omega_n$.

(& not $\omega_{n+1}, \dots \omega_N$)

- ▷ Equivalently, for any $B \subseteq \mathbb{R}$, the event $\{X \in B\} \in \mathcal{F}_n$.
- ▷ Equivalently, if $\omega' \in \Pi_n(\omega)$ then $X(\omega') = X(\omega)$.

Question 5.22. *Let $X(\omega) \stackrel{\text{def}}{=} \omega_1 - 10\omega_2$. For what n is $\mathcal{F}_n$-measurable?*

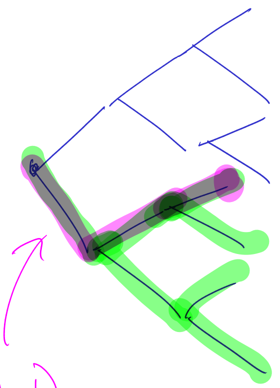
Please enable video if possible.

Office Hours today: End at 5:00 PM.

$\Omega \longrightarrow \{-1, 1\}^N$ (Sample space → N coin tosses)

$\omega \in \Omega$. $\omega = (\omega_1, \ldots \omega_N) \longrightarrow \omega_i \in \{-1, 1\} \rightarrow$ outcome of i^{th} coin toss

Visualize as trees. $\omega = (-1, 1, 1) \in \Omega$. $(N = 3)$



$n \in \{0, \dots N\}$

$\Pi_n(\omega)$ = {all sample points ω' for which the first n coin tosses agree with those in ω}.

i.e. $\Pi_n(\omega) = \{\omega' \in \Omega \mid \omega' = (\omega'_1, \dots \omega'_N)$
$\& \ \omega'_1 = \omega_1, \ \omega'_2 = \omega_2 \dots$
$\omega'_n = \omega_n \}$.

$n = 1$. Draw $\Pi_1(\omega)$

$\omega = (-1, 1, 1)$

5.2. Filtrations.

- Let $N \in \mathbb{N}$, $d_1, \dots, d_N \in \mathbb{N}$, $\Omega = \{1, \dots, d_1\} \times \{1, \dots, d_n\} \times \cdots \times \{1, \dots, d_N\}$.
- That is $\Omega = \{\omega \mid \omega = (\omega_1, \dots, \omega_N),\ \omega_i \in \{1, \dots, d_i\}\}$.
- $d_n = 2$ for all n corresponds to flipping a two sided coin at every time step.

Definition 5.18. We define a *filtration* on Ω as follows:

▷ $\mathcal{F}_0 = \{\emptyset, \Omega\}$.
▷ $\mathcal{F}_1 =$ all events that can be described by only the first coin toss (die roll). E.g. $A = \{\omega \mid \omega_1 = H\} \in \mathcal{F}_1$.
▷ $\mathcal{F}_n =$ all events that can be described by only the first n coin tosses.

More precisely, given $\omega = (\omega_1, \dots, \omega_N) \in \Omega$ and $n \in \{0, \dots, N\}$ define

$$\Pi_n(\omega) = \{\omega' \in \Omega \mid \omega' = (\omega'_1, \dots, \omega'_N) \text{ and } \omega'_i = \omega_i \text{ for all } i \leqslant n\}.$$

Now $\mathcal{F}_n$ is defined by $\mathcal{F}_n \overset{\text{def}}{=} \left\{A \subseteq \Omega \,\middle|\, A = \bigcup_{i=1}^{k} \Pi_n(\omega^i),\ \omega^1, \dots, \omega^k \in \Omega\right\}$

Remark 5.19. Note $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_N = \mathcal{P}(\Omega)$.

Question 5.20. *Let $\Omega = \{H, T\}^3 \cong \{1, 2\}^3$. What are $\mathcal{F}_0, \dots, \mathcal{F}_3$?*

↳ $\Omega = \{-1, 1\}^3$

($-1 \to$ tails
$+1 \to$ Heads)

$N = 3$

Compute $\mathcal{F}_0 = \{\phi, \Omega\}$

$$\mathcal{F}_1 = \{\phi, \Omega, \{(1,1,1),(1,1,-1),(1,-1,1),(1,-1,-1)\}, \{(-1,1,1),(-1,1,-1),(-1,-1,1),(-1,-1,-1)\}\}$$

$$\mathcal{F}_2 = \{\phi, \Omega, \{(1,1,1),(1,1,-1)\}, \{(1,1,1),(1,1,-1),(1,-1,1),(1,-1,-1)\},$$

$\Pi_2((1,1,1))$

$\Pi_2((1,1,1)) \cup \Pi_2((1,-1,1))$

& keep going }.

$$\Pi_2(1,1,1) = \{(1,1,-1),\ (1,1,1)\}$$

$$\Pi_2((1,-1,1)) = \{(1,-1,1),\ (1,-1,-1)\}$$

Definition 5.21. Let $n \in \{0, \dots, N\}$. We say a random variable X is $\mathcal{F}_n$-measurable if $X(\omega)$ only depends on $\omega_1, \dots, \omega_n$.

- ▷ Equivalently, for any $B \subseteq \mathbb{R}$, the event $\{X \in B\} \in \mathcal{F}_n$.
- ▷ Equivalently, if $\omega' \in \Pi_n(\omega)$ then $X(\omega') = X(\omega)$.

$\{X \in B\} = \{\omega \in \Omega \mid X(\omega) \in B\}$

Question 5.22. *Let $X(\omega) \stackrel{\text{def}}{=} \omega_1 - 10\omega_2$. For what n is $\mathcal{F}_n$-measurable?*

$X(\omega) = \omega_1 - 10\omega_2$ is $\mathcal{F}_n$ meas $\forall n \geq 2$

In finance: Always require trading strategies to be "adapted"

i.e. Δ_n = # shares of stock in your Pf at time n.

$\boxed{\text{Always need } \Delta_n \text{ to be } \mathcal{F}_n \text{ measurable.}}$

X is $\mathcal{F}_0$ meas $\iff$ X is constant.

Cond Prob $\implies P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

Diff from $\mathbf{E}$

5.3. Conditional expectation.

Definition 5.23. Let X be a random variable, and $n \leqslant N$. We define $\boldsymbol{E}(X \mid \mathcal{F}_n) = \boldsymbol{E}_n X$ to be the *random variable* given by

$$(E_n X)(\omega) = \boldsymbol{E}_n X(\omega) = \frac{\sum_{\omega' \in \Pi_n(\omega)} p(\omega') X(\omega')}{\sum_{\omega' \in \Pi_n(\omega)} p(\omega')}, \qquad \text{where} \quad \boxed{\Pi_n(\omega)} = \{\omega' \in \Omega \mid \omega'_1 = \omega_1, \ldots, \omega'_n = \omega_n\}$$

Remark 5.24. $\boldsymbol{E}_n X$ is the "best approximation" of X given only the first n coin tosses.

Remark 5.25. The above formula does not generalize well to infinite probability spaces. We will develop a definition that does generalize; after we have that definition we will never ever ever use this formula.

$E_n X = E(X \mid \mathcal{F}_n) =$ condition expectation of X given $\mathcal{F}_n$.

given

NOTE: $E_n X$ is a $\mathcal{F}_n$-meas Random Variable

Note $E_n X(\omega) =$ avg of X over the event $\Pi_n(\omega) = \frac{1}{P(\Pi_n(\omega))} \sum_{\omega' \in \Pi_n(\omega)} p(\omega') X(\omega')$

Proposition 5.26. *The conditional expectation $\boldsymbol{E}_n X$ defined by the above formula satisfies the following two properties:*

(1) $\boldsymbol{E}_n X$ *is an $\mathcal{F}_n$-measurable* random variable.

(2) *For every $A \in \mathcal{F}_n$,* $\displaystyle\sum_{\omega\in A} \boldsymbol{E}_n X(\omega)p(\omega) = \sum_{\omega \in A} X(\omega)p(\omega).$

Choose $A = \Omega$. Then $\boldsymbol{E}X = \sum_{\omega \in A} X(\omega) p(\omega)$.

By (2) also have $\boldsymbol{E}X = \sum_{\omega \in A} \boldsymbol{E}_n X(\omega)\, p(\omega) = \boldsymbol{E}(\boldsymbol{E}_n X)$

$\Rightarrow \boldsymbol{E}(\boldsymbol{E}_n X) = \boldsymbol{E}X$

Lecture 6 (9/13). Please enable your video if possible.

(1) Finance use of cond exp:

Given any security with payoff V_N at time N.

(For nice markets) AFP at time $n \leq N$

is $\boxed{\text{Conditional exp}}$ of V_N (wrt the "Risk Neutral measure")

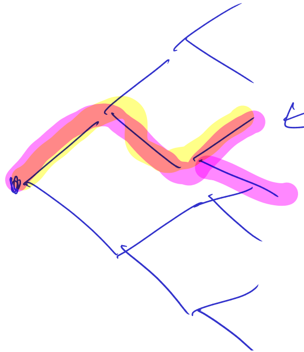
& discounted

4

Recall: $\mathcal{F}_n$ = all events describable by <u>only</u> first n coins

$$= \left\{ A \subseteq \Omega \;\middle|\; A = \bigcup_{i=1}^{k} \Pi_n(\omega^i) \text{ for some } k \in \mathbb{N} \ \& \ \omega^1, \dots, \omega^k \in \Omega \right\}.$$

$$\Pi_n(\omega) = \{ \omega' \in \Omega \mid \omega'_i = \omega_i \ \forall i \leq n \}$$



ω = Yellow highlighted path.

$\Pi_2(\omega)$ = pink highlighted things.

5.3. Conditional expectation.

Definition 5.23. Let X be a random variable, and $n \leqslant N$. We define $\boldsymbol{E}(X \mid \mathcal{F}_n) = \boldsymbol{E}_n X$ to be the *random variable* given by

$$\boldsymbol{E}_n X(\omega) = \frac{\displaystyle\sum_{\omega' \in \Pi_n(\omega)} p(\omega') X(\omega')}{\displaystyle\sum_{\omega' \in \Pi_n(\omega)} p(\omega')}, \qquad \text{where} \qquad \Pi_n(\omega) = \{\omega' \in \Omega \mid \omega'_1 = \omega_1, \ldots, \omega'_n = \omega_n\}$$

Remark 5.24. $\boldsymbol{E}_n X$ is the "best approximation" of X given only the first n coin tosses.

Remark 5.25. The above formula does not generalize well to infinite probability spaces. We will develop a definition that does generalize; after we have that definition we will never ever ever use this formula.

$E_n X(\omega) =$ Average of X on the event $\Pi_n(\omega)$

$$= \frac{1}{P(\Pi_n(\omega))} \underbrace{\sum_{\omega' \in \Pi_n(\omega)} X(\omega')\, p(\omega')}.$$

Proposition 5.26. *The conditional expectation $\boldsymbol{E}_n X$ defined by the above formula satisfies the following two properties:*

1. $\boldsymbol{E}_n X$ *is an $\mathcal{F}_n$-measurable* random variable.
2. *For every* $A \in \mathcal{F}_n$, $\displaystyle\sum_{\omega\in A} \boldsymbol{E}_n X(\omega)p(\omega) = \sum_{\omega\in A} X(\omega)p(\omega)$.

Note (2) $\to$ Average of $E_n X$ on any $\mathcal{F}_n$-meas event A

= Average of X " the same event A.

Note: $A \in \mathcal{F}_n$. Average of $E_n X$ on A $= \dfrac{1}{P(A)} \displaystyle\sum_{\omega' \in A} E_n X(\omega') p(\omega')$

" Average of X on A $= \dfrac{1}{P(A)} \displaystyle\sum_{\omega' \in A} X(\omega') p(\omega')$

Pf of ①: NTS $\widetilde{E_n X}$ is $\mathcal{F}_n$ meas.

i.e. NTS: If $\tilde{\omega} \in \Omega$ is such that $\tilde{\omega}_i = \omega_i \ \forall i \leq n$

then $E_n X(\tilde{\omega}) = E_n X(\omega)$

Pf: Note: If $\tilde{\omega}$ is as above

$$\Pi_n(\tilde{\omega}) = \Pi_n(\omega)$$

Hence $E_n X(\tilde{\omega}) = \dfrac{1}{P(\Pi_n(\tilde{\omega}))} \displaystyle\sum_{\omega' \in \Pi_n(\tilde{\omega})} X(\omega')\, p(\omega')$

$$= \frac{1}{P(\Pi_n(\omega))} \sum_{\omega' \in \Pi_n(\omega)} X(\omega')\, p(\omega') = E_n X(\omega)$$

QED.

Proof of (2):

(1) For any $\omega \in \Omega$, $\displaystyle\sum_{\omega' \in \Pi_n(\omega)} \boldsymbol{E}_n X(\omega') p(\omega') = \sum_{\omega' \in \Pi_n(\omega)} X(\omega') p(\omega')$

Proof (2) NTS. $\forall A \in \mathcal{F}_n$, $\displaystyle\sum_{\omega' \in A} E_n X(\omega') p(\omega') = \sum_{\omega' \in A} X(\omega') p_n(\omega')$.

Step (1) above is checking this for $A = \Pi_n(\omega)$.

Note $\forall \omega' \in \Pi_n(\omega)$, $E_n X(\omega') = E_n X(\omega)$ (by part 1).

$\Rightarrow$ LHS $= \displaystyle\sum_{\omega' \in \Pi_n(\omega)} E_n X(\omega') p(\omega')$

$$= \sum_{\omega' \in \Pi_n(\omega)} E_n X(\omega)\, p(\omega') = E_n X(\omega)\, P(\Pi_n(\omega))$$

by formula for $E_n X$

$$\sum_{\omega' \in \underbrace{\Pi_n(\omega)}} X_n(\omega')\, p(\omega') = \text{RHS}$$

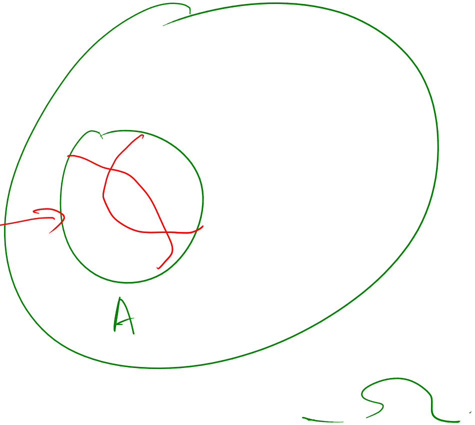
QED.

(2) For any A in Ω, then there exist $\omega^1, \dots, \omega^k \in \Omega$ such that A is the *disjoint union* of $\Pi_n(\omega^1), \dots, \Pi_n(\omega^k)$.

$A \in \mathcal{F}_n$

Say (for picture)

$A = \Pi_n(\omega^1) \cup \Pi_n(\omega^2)$



Obs: $\forall\, \omega^1, \omega^2 \in \Omega$.

Either (1) $\Pi_n(\omega^1) = \Pi_n(\omega^2)$ OR (2) $\Pi_n(\omega^1) \cap \Pi_n(\omega^2) = \phi$

(3) Hence $\sum_{\omega\in A} \boldsymbol{E}_n X(\omega)p(\omega) = \sum_{i=1}^{k} \sum_{\omega\in\Pi_n(\omega^i)} \boldsymbol{E}_n X(\omega)p(\omega) = \sum_{i=1}^{k} \sum_{\omega\in\Pi_n(\omega^i)} X(\omega)p(\omega) = \sum_{\omega\in A} X(\omega)p(\omega)$.

QED

Know ① $\forall \omega$, $\sum_{\omega'\in\Pi_n(\omega)} E_n X(\omega') p(\omega') = \sum_{\omega'\in\Pi_n(\omega)} X(\omega') p(\omega')$,

② $A \in \mathcal{F}_n$ is of the form $A = \underbrace{\bigcup_{i=1}^{k} \Pi_n(\omega^i)}$ disjoint union.

Proposition 5.27 (Uniqueness)**.** *If Y and Z are two $\mathcal{F}_n$-measurable random variables such that $\sum_{\omega \in A} Y(\omega)p(\omega) = \sum_{\omega \in A} Z(\omega)p(\omega)$ for every $A \in \mathcal{F}_n$, then we must have $\boldsymbol{P}(Y = Z) = 1$.*

IOU Proof (Next time)

Definition 5.28. Let X be a random variable, and $n \leqslant N$. We define the *conditional expectation of X given $\mathcal{F}_n$*, denoted by $\boldsymbol{E}_n X$, or $\boldsymbol{E}(X \mid \mathcal{F}_n)$, to be the unique *random variable* such that:

(1) $\boldsymbol{E}_n X$ is a $\mathcal{F}_n$-measurable random variable.
(2) For every $A \subseteq \mathcal{F}_n$, we have $\sum_{\omega \in A} \boldsymbol{E}_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$.

Remark 5.29. This is the definition that generalizes to the continuous case. All properties we develop on conditional expectations will only use the above definition, and not the explicit formula.

Note: 1 We have checked the the formula for $E_n X$ above satisfies both conditions 1 & 2

& 2 Uniqueness: by Prop 5.27.

Lecture 7 (9/15). Please enable your video if you can.

Last time: $E_n X(\omega)$ = cond. exp of X given $\mathcal{F}_n$

$$= \frac{1}{P(\Pi_n(\omega))} \sum_{\omega' \in \Pi_n(\omega)} X(\omega') p(\omega')$$

= Average of X on the event $\Pi_n(\omega)$

= Avg of X ~~on~~ over all possible future coin tosses given that the first n have come up $\omega_1, \omega_2 \cdots \omega_n$.

Proposition 5.27 (Uniqueness). *If Y and Z are two $\mathcal{F}_n$-measurable random variables such that $\sum_{\omega\in A} Y(\omega)p(\omega) = \sum_{\omega\in A} Z(\omega)p(\omega)$ for every $A \in \mathcal{F}_n$, then we must have $\boldsymbol{P}(Y = Z) = 1$.*

Note avg of Y on $A = \dfrac{1}{P(A)} \sum_{\omega\in A} Y(\omega)p(\omega)$

" " Z on $A = \dfrac{1}{P(A)} \sum_{\omega\in A} Z(\omega)p(\omega)$

equal $\Leftarrow$

Pf: Let $A = \{Y > Z\} = \{\omega\in\Omega \mid Y(\omega) > Z(\omega)\}$.

Q1: $A \in \mathcal{F}_n$? (Yes because Y & Z are both $\mathcal{F}_n$ meas).

$\Rightarrow$ By assumption $\sum_{\omega\in A} Y(\omega)p(\omega) = \sum_{\omega\in A} Z(\omega)p(\omega)$

Note $\forall \omega \in A$ $\overline{Y(\omega) > Z(\omega)}$

Only possible if $f(\omega) = 0$ $\forall \omega \in A$ (or $\underline{A = \phi}$)

$\Rightarrow P(A) = 0$

$$\left.\begin{array}{l} \Rightarrow P(Y > Z) = 0 \\ \text{Similarly can show } P(Y < Z) = 0 \end{array}\right\} \begin{array}{l} \Rightarrow P(Y \neq Z) = 0 \\ \Leftrightarrow P(Y = Z) = 1 \end{array}$$

QED.

Definition 5.28. Let X be a random variable, and $n \leqslant N$. We define the *conditional expectation of X given $\mathcal{F}_n$*, denoted by $\boldsymbol{E}_n X$, or $\boldsymbol{E}(X \mid \mathcal{F}_n)$, to be the unique *random variable* such that:

(1) $\boldsymbol{E}_n X$ is a $\mathcal{F}_n$-measurable random variable.

(2) For every $A \subseteq \mathcal{F}_n$, we have $\sum_{\omega \in A} \boldsymbol{E}_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$.

Remark 5.29. This is the definition that generalizes to the continuous case. All properties we develop on conditional expectations will only use the above definition, and not the explicit formula.

avg of $E_n X$ on A = avg. of X on A

$\forall A \in \mathcal{F}_n$

Note (a) the formula for $E_n X$ given above certainly satisfies (1) & (2) (last time)

(b) Uniqueness: follows from 5.27.

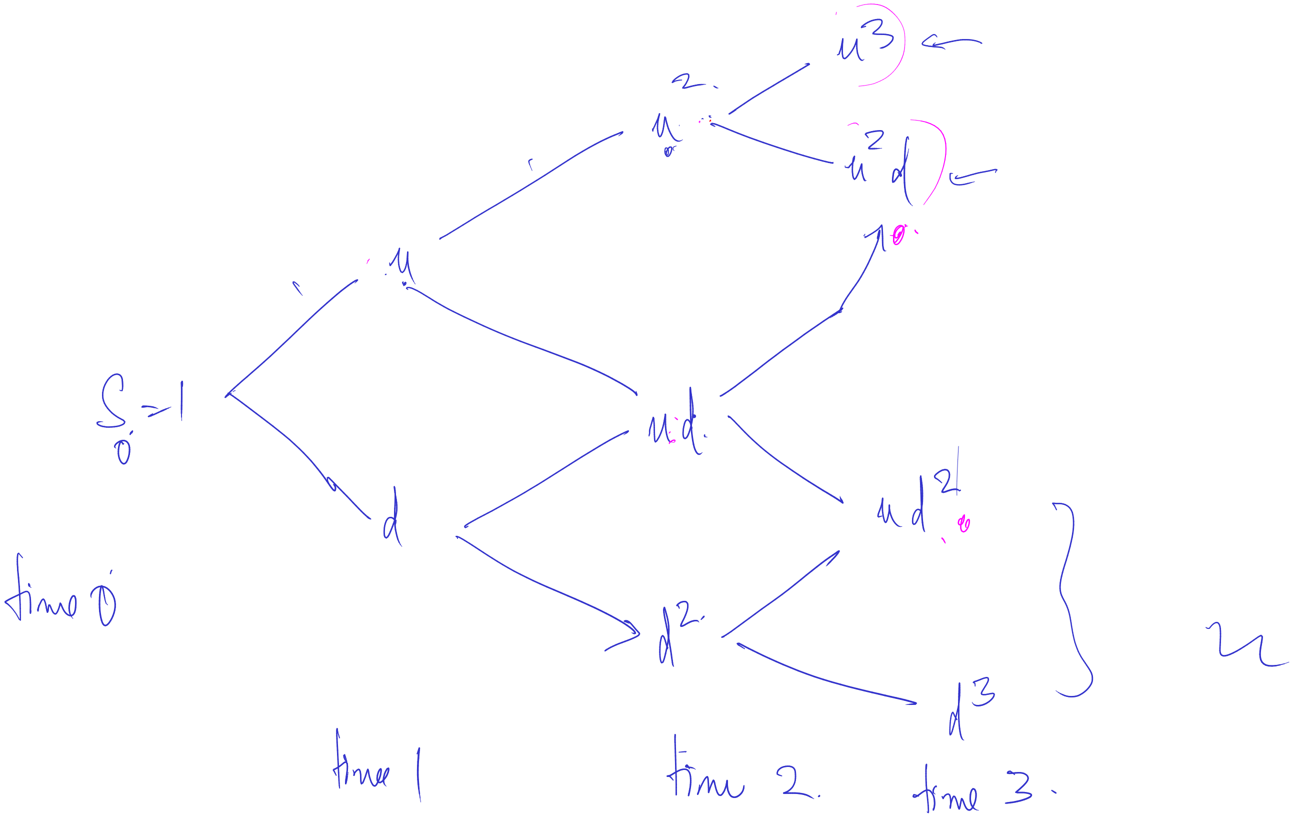
Example 5.30. Let S_n be the stock price in the binomial model after n periods. Compute $E_1 S_3$, $E_2 S_3$.

Flip a coin at every time step $\begin{cases} \to \text{Heads with prob } p \\ \to \text{Tails with prob } q = 1-p. \end{cases}$

$$S_{n+1} = \begin{cases} u S_n & \text{if } n+1^{th} \text{ coin flips heads} \\ d S_n & \text{if } (n+1)^{th} \text{ coin flips tails.} \end{cases}$$

$S_0 = 1.$

$u \quad n$



S0 = 1
u
d
u2
ud
d2
u3
u2d
ud2
d3
time 0
time 1
time 2
time 3
n

Find $E_2 S_3$. $\omega = (\omega_1, \omega_2, \omega_3)$ $\omega_i \in \{\pm 1\}$

$\hookrightarrow E_2 S_3$ is an $\mathcal{F}_2 \to$ meas RV.

$\omega = (1, 1, *)$

$\Pi_2(\omega) = \{(1,1,-1), (1,1,+1)\}$.

$P(\Pi_2(\omega)) = p^2 q + p^3 = p^2$

$E_2 S_3(\omega) = E_2 S_3(\omega_1, \omega_2, \omega_3)$ does not dep on ω_3

① Find $\to E_2 S_3(1, 1, *) = u^3 \cdot p + u^2 d \cdot q = \dfrac{p^3 u^3 + p^2 q u^2 d}{p^2}$.

$\to E_2 S_3(1, -1, *) = u^2 d p + u d^2 q$.

$$\to \quad E_2 S_3(-1, 1, x) = u^2 d p + u d^2 q$$

$$\to \quad E_2 S_3(-1, -1, x) = u d^2 p + d^3 \cdot q.$$

Claim: $\quad E_2 S_3 = (pu + qd) S_2$

Theorem 5.31. *(1) If X, Y are two random variables and $\alpha \in \mathbb{R}$, then $\boldsymbol{E}_n(X + \alpha Y) = \boldsymbol{E}_n X + \alpha \boldsymbol{E}_n Y$. (On homework)*
(2) (Tower property) *If $m \leqslant n$, then $\boldsymbol{E}_m(\boldsymbol{E}_n X) = \boldsymbol{E}_m X$.*
(3) If X is $\mathcal{F}_n$ measurable, and Y is any random variable, then $\boldsymbol{E}_n(XY) = X \boldsymbol{E}_n Y$.

X & Y 2RV's : $E(XY) \neq X \underbrace{EY}_{\text{RV.}}$

$E_n(XY) = X \, E_n Y$

Say X is $\mathcal{F}_n$ - meas.

m

Pf of Tower: NTS $m \le n$. then $\underbrace{E_m(\underline{E_n X})} = \underbrace{E_m X}$.

(1) Will show (1) $E_m(E_n X)$ is $\mathcal{F}_m$ meas. ← True because $E_m(\text{anything})$ is $\mathcal{F}_m$ meas.

& (2) $\forall A \in \mathcal{F}_m$, $\sum_{\omega \in A} E_m(E_n X)(\omega)\, p(\omega) = \sum_{\omega \in A} X(\omega)\, p(\omega)$.

Pf: $A \in \mathcal{F}_m$, $\sum_{\omega \in A} E_m(E_n X)(\omega)\, p(\omega) \overset{\text{Def of } E_m}{=} \sum_{\omega \in A} \underbrace{E_n X(\omega)}\, p(\omega)$

m $\overline{A \in \mathcal{F}_n}$, $\sum_{\omega \in A} X(\omega)\, p(\omega)$ QED.

Def of E_m : $\forall\, A \in \mathcal{F}_{\underline{\underline{m}}}$, & any RV Y,

$$\sum_{\omega \in A} E_{\underline{\underline{m}}} Y(\omega)\, p(\omega) = \sum_{A} Y(\omega)\, p(\omega)$$

Theorem 5.32. *If X is independent of $\mathcal{F}_n$ then $\boldsymbol{E}_n X = \boldsymbol{E} X$.*

Theorem 5.33 (Independence lemma)**.** *If X is independent of $\mathcal{F}_n$ and Y is $\mathcal{F}_n$-measurable, and $f\colon \mathbb{R} \to \mathbb{R}$ is a function then*

$$\boldsymbol{E}_n f(X, Y) = \sum_{i=1}^{m} f(x_i, Y)\boldsymbol{P}(X = x_i)\,, \qquad \textit{where } \{x_1, \dots, x_m\} = X(\Omega)\,.$$

5.4. **Martingales.**

Definition 5.34. A *stochastic process* is a collection of random variables X_0, X_1, $\ldots$, X_N.

Definition 5.35. A stochastic process is *adapted* if X_n is $\mathcal{F}_n$-measurable for all n. (Non-anticipating.)

Question 5.36. *Is $X_n(\omega) = \sum_{i \leqslant n} \omega_i$ adapted?*

Question 5.37. *Is $X_n(\omega) = \omega_n$ adapted? Is $X_n(\omega) = 15$ adapted? Is $X_n(\omega) = \omega_{15}$ adapted? Is $X_n(\omega) = \omega_{N-i}$ adapted?*

Remark 5.38. We will always model the price of assets by *adapted* processes. We will also only consider trading strategies which are adapted.

Example 5.39 (Money market). Let $Y_0 = Y_0(\omega) = a \in \mathbb{R}$. Define $Y_{n+1} = (1+r)Y_n$. (Here r is the interest rate.)

Example 5.40. Suppose $\Omega = \{\pm 1\}^N \cong \{H, T\}^N \cong \{1, 2\}^N$. Let $S_0 = a \in \mathbb{R}$. Define $S_{n+1}(\omega) = \begin{cases} uS_n(\omega) & \omega_{n+1} = 1\,, \\ dS_n(\omega) & \omega_{n+1} = -1\,. \end{cases}$

Is S_n adapted? (Used to model stock price in the multi-period Binomial model.)

Definition 5.41. We say an adapted process M_n is a martingale if $\boldsymbol{E}_n M_{n+1} = M_n$. (Recall $\boldsymbol{E}_n Y = \boldsymbol{E}(Y \mid \mathcal{F}_n)$.)

Remark 5.42. Intuition: A martingale is a "fair game".

Example 5.43 (Unbiased random walk). If $X_1, \dots, X_N$ are i.i.d. and mean zero, then $S_n = \sum_{k=1}^n X_k$ is a martingale.

Question 5.44. *If M is a martingale, and $m \leqslant n$, is $\boldsymbol{E}_m M_n = M_m$?*

Question 5.45. *If M is a martingale does $\boldsymbol{E}M_n$ change with n?*

Question 5.46. *Conversely, if $\boldsymbol{E} M_n$ is constant, is M a martingale?*

Lecture 8 (9/17). Please enable your video if you can.

Last time: $X \to$ R.V., $n \le N$. $E_n X$ = cond exp of X given $\mathcal{F}_n$.

$E_n X$ is the unique RV such that

(1) $E_n X$ is $\mathcal{F}_n$ meas

& (2) $\forall A \in \mathcal{F}_n$, $\sum_{\omega \in A} E_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$

Theorem 5.31. *(1) If X, Y are two random variables and $\alpha \in \mathbb{R}$, then $\boldsymbol{E}_n(X + \alpha Y) = \boldsymbol{E}_n X + \alpha \boldsymbol{E}_n Y$. (On homework).*
(2) (Tower property) *If $m \leqslant n$, then $\boldsymbol{E}_m(\boldsymbol{E}_n X) = \boldsymbol{E}_m X$.*
(3) If X is $\mathcal{F}_n$ measurable, and Y is any random variable, then $\boldsymbol{E}_n(XY) = X\boldsymbol{E}_n Y$.

(2) last time

Pf of (3): To show $X E_n Y = E_n(XY)$ we only NTS

(a) $X E_n Y$ is $\mathcal{F}_n$ - meas.

(b) $\forall A \in \mathcal{F}_n$, $\quad \sum_{\omega \in A} X(\omega) E_n Y(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) Y(\omega) p(\omega)$

Pf of (a): X is $\mathcal{F}_n$ - meas by assumption
$E_n Y$ " " " by def
} product $\Rightarrow X E_n Y$ is $\mathcal{F}_n$ meas

(b) Let $\{x_1, \dots x_k\} =$ Range of X.

Write $A = \bigcup_{i=1}^{k} \{X = x_i\} \cap A$ (disj union)

$$\sum_{\omega \in A} X(\omega) E_n Y(\omega) \phi(\omega) = \sum_{i=1}^{k} \sum_{\omega \in \underbrace{A \cap \{X = x_i\}}} X(\omega) E_n Y(\omega) \phi(\omega)$$

$$= \sum_{i=1}^{k} \sum_{\omega \in A \cap \{X = x_i\}} x_i E_n Y(\omega) \phi(\omega)$$

$$= \sum_{i=1}^{k} x_i \sum_{\omega \in A \cap \{X = x_i\}} E_n Y(\omega) \phi(\omega).$$

$$= \sum_{i=1}^{k} x_i \sum_{\omega \in A \cap \{X = x_i\}} Y(\omega) p(\omega) \quad \left(\because \underline{A} \cap \{X = x_i\} \text{ is } \mathcal{F}_n \text{ meas} \right)$$

$$= \underbrace{\sum_{i=1}^{k} \sum_{\omega \in A \cap \{X = x_i\}}} \overset{\to X(\omega)}{x_i} Y(\omega) p(\omega) \quad \left(\because X \text{ is } \mathcal{F}_n\text{-meas} \right).$$

$$= \sum_{\omega \in A} X(\omega) Y(\omega) p(\omega)$$

QED.

Theorem 5.32. *If X is independent of $\mathcal{F}_n$ then $\boldsymbol{E}_n X = \boldsymbol{E} X$.*

(i.e. for every $A \in \mathcal{F}_n$ & $B \subseteq \mathbb{R}$, the events A & B are ind)

($\iff$ X is ind of $\mathcal{F}_n$, if $\text{Range}(X) = \{x_1, \dots x_k\}$
& $\forall i \in \{1, \dots k\}$ $\forall A \in \mathcal{F}_n$, A & $\{X = x_i\}$ are ind)

Pf: Assume X ind of $\mathcal{F}_n$.

NTS $E_n X = EX$.

i.e. NTS ① EX is $\mathcal{F}_n$ meas -

(b) $\forall A \in \mathcal{F}_n$, $\underbrace{\sum_{\omega \in A} \mathbb{E}X \, p(\omega)} = \underbrace{\sum_{\omega \in A} \underline{X}(\omega) \, p(\omega)}$.

Pf of (a): Note $\mathbb{E}X$ is a const R.V. (ie $\forall \omega \in \Omega$, the R.V. takes on the value $\mathbb{E}X \in \mathbb{R}$).

$\mathbb{E}X$ is $\mathcal{F}_n$ meas $\forall n$. ✓

Pf of (b): Start with RHS. Let $\text{Range}(X) = \{x_1, \dots x_k\}$

$$A = \bigcup_{i=1}^{k} A \cap \{X = x_i\}.$$

$$\sum_{\omega\in A} X(\omega)\,p(\omega) = \sum_{i=1}^{k} \sum_{\omega\in A\cap\{X=x_i\}} X(\omega)\,p(\omega)$$

$$= \sum_{i=1}^{k} x_i \sum_{\omega\in A\cap\{X=x_i\}} p(\omega)$$

$$= \sum_{i=1}^{k} x_i\, P(A\cap\{X=x_i\})$$

$$= \sum_{i=1}^{k} x_i\, P(A)\,P(X=x_i)$$

($\because X$ is ind of $\mathcal{E}_n$ $\Rightarrow A$ & $\{X=x_i\}$ are ind.)

$$= P(A) \sum_{i=1}^{k} x_i P(X = x_i)$$

$$= P(A) \cdot EX = \sum_{\omega \in A} EX \, p(\omega).$$

$\boxed{\text{QED.}}$

Intuition so far: Computing E_n

- $\mathcal{E}_n$ mess things $\rightarrow$ leave alone
- $\mathcal{E}_n \rightarrow$ indep things $\rightarrow$ average

Theorem 5.33 (Independence lemma). *If X is independent of $\mathcal{F}_n$ and Y is $\mathcal{F}_n$-measurable, and $f : \mathbb{R} \to \mathbb{R}$ is a function then*

$$\boldsymbol{E}_n f(X, Y) = \sum_{i=1}^{m} f(x_i, Y) \boldsymbol{P}(X = x_i) , \qquad \textit{where } \{x_1, \dots, x_m\} = X(\Omega) .$$

On HW.

Range $(X) = \{x_1, \dots, x_m\}$, $\boxed{y \in \mathbb{R}}$ (a real #)

$$E f(X, y) = \sum_{i=1}^{m} f(x_i, y) P(X = x_i)$$

5.4. Martingales.

Definition 5.34. A *stochastic process* is a collection of random variables $X_0, X_1, \dots, X_N$.

Definition 5.35. A stochastic process is *adapted* if X_n is $\mathcal{F}_n$-measurable for all n. (Non-anticipating.)

Question 5.36. *Is* $X_n(\omega) = \sum_{i \leqslant n} \omega_i$ *adapted?*

Question 5.37. *Is* $X_n(\omega) = \omega_n$ *adapted? Is* $X_n(\omega) = 15$ *adapted? Is* $X_n(\omega) = \omega_{15}$ *adapted? Is* $X_n(\omega) = \omega_{N-i}$ *adapted?*

Remark 5.38. We will always model the price of assets by *adapted* processes. We will also only consider trading strategies which are adapted.

$$X_n(\omega) = \sum_{i=1}^{n} \omega_i \qquad (\omega_i \in \{\pm 1\}.)$$

1. $\to$ Stochastic process? Yes

2. X_n adapted?

$$\left.\begin{aligned} X_1(\omega) &= \omega_1 \leftarrow \mathcal{F}_1 \text{ meas} \\ X_2(\omega) &= \omega_1 + \omega_2 \leftarrow \mathcal{F}_2 \text{ meas} \\ X_3(\omega) &= \omega_1 + \omega_2 + \omega_3 \leftarrow \mathcal{F}_3 \text{ meas} \end{aligned}\right\} \text{ works then.}$$

Example 5.39 (Money market). Let $Y_0 = Y_0(\omega) = a \in \mathbb{R}$. Define $Y_{n+1} = (1+r)Y_n$. (Here r is the interest rate.)

Example 5.40. Suppose $\Omega = \{\pm 1\}^N \cong \{H, T\}^N \cong \{1, 2\}^N$. Let $S_0 = a \in \mathbb{R}$. Define $S_{n+1}(\omega) = \begin{cases} uS_n(\omega) & \omega_{n+1} = 1\,, \\ dS_n(\omega) & \omega_{n+1} = -1\,. \end{cases}$ Is S_n adapted? (Used to model stock price in the multi-period Binomial model.)

Definition 5.41. We say an adapted process M_n is a martingale if $\boxed{\boldsymbol{E}_n M_{n+1} = M_n.}$ (Recall $\boldsymbol{E}_n Y = \boldsymbol{E}(Y \mid \mathcal{F}_n)$.)

Remark 5.42. Intuition: A martingale is a "fair game".

Example 5.43 (Unbiased random walk). If $X_1, \dots, X_N$ are i.i.d. and mean zero, then $S_n = \sum_{k=1}^n X_k$ is a martingale.

Lecture 8 (9/20). Please Enable Your Video If you Can

Last time: Stochastic process ~~is~~ X_n is a RV $\forall n$.

$\boxed{\text{Adapted}}$: $\forall n$, need X_n to be $\mathcal{F}_n$ meas.

(i.e. X_1 is $\mathcal{F}_1$ meas
X_2 " $\mathcal{F}_2$ "
etc.

All trading strats & stock prices etc.

Intuition : Martingale $\longrightarrow$ "Fair Game"

$M_n \longrightarrow$ adapted Stochastic process.

$\hookrightarrow$ "Winnings at time n after playing a game"

Q : Walk away at time n with \$ M_n in hand.

Play once more $\longleftarrow$ $\boxed{\text{Keep playing if game is "fair"}}$

2

At time n say first n coins came up

$\omega_1, \omega_2 \cdots \omega_n$.

Let $\omega = (\omega_1, \omega_2 \cdots \omega_n, \underbrace{* \quad * \quad \cdots},)$

Cash in hand for this seq at time $n = M_n(\omega)$

Expected return if I play one more, given the first n.

coins are $(\omega_1, \cdots \omega_n)$: $\boxed{E_n M_{n+1}}$

u

If $\underbrace{E_n M_{n+1} = M_n}$ $\leftarrow$ Game is fair!

Definition 5.41. We say an adapted process M_n is a martingale if $\boldsymbol{E}_n M_{n+1} = M_n$. (Recall $\boldsymbol{E}_n Y = \boldsymbol{E}(Y \mid \mathcal{F}_n)$.)

Remark 5.42. Intuition: A martingale is a "fair game".

Example 5.43 (Unbiased random walk). If $X_1, \dots, X_N$ are i.i.d. and mean zero, then $S_n = \sum_{k=1}^n X_k$ is a martingale.

indep & identically dist.

$E X_i = 0$

Eg: $X_n = \begin{cases} +1 & \text{if } n^{th} \text{ coin flips heads} \\ -1 & \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{tails} \end{cases}$ (iid & $P(\text{heads}) = P(\text{tails}) = \frac{1}{2}$)

S_n = cumulative winnings after time n.

Intuition $\longrightarrow$ seems like a fair game.

Math: $E_n(S_{n+1}) \overset{\text{NTS}}{=} S_n$

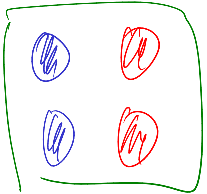
$$E_n(S_{n+1}) = E_n(S_n + X_{n+1})$$

$$= E_n S_n + E_n X_{n+1}$$

$$= S_n + \underbrace{E X_{n+1}}_{0 \text{ by assumption.}} \quad (\because X_{n+1} \text{ is ind of } X_n)$$

$\Rightarrow E_n S_{n+1} = S_n \Rightarrow S$ is a mg.

Example 5.44 (Drawing balls without replacement). Red or Blue balls are drawn from a container *without replacement*. The container has 2 red and 2 balls initially. You win \$1 if the ball is blue, and lose \$1 if the ball is red. Is the process of your winnings a martingale?



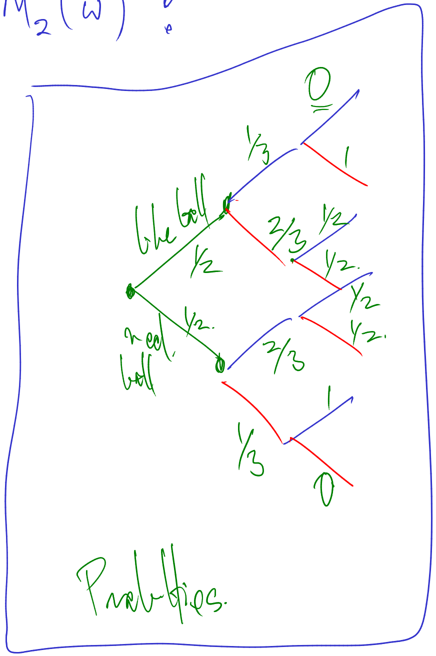
Guess: Not a Mg.

Compute $E_1 M_2$ → M_n = $\underline{\text{total}}$ winnings after time n.

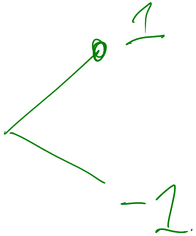
$$M_1(\omega) = \begin{cases} 1 & \omega_1 = \text{blue} \\ -1 & \omega_1 = \text{red} \end{cases}$$

2

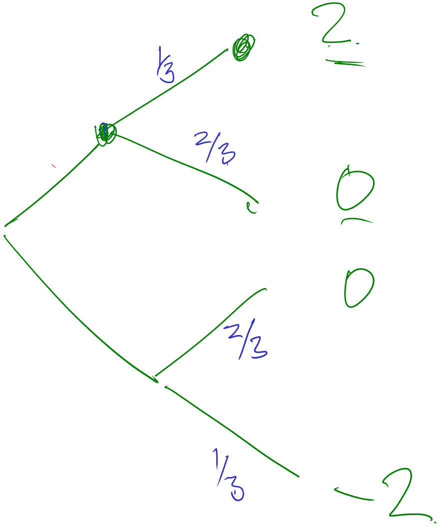
$M_2(\omega)$:



M_1



M_2.



Compute $E_1 M_2$?

$2\left(\frac{1}{3}\right) + 0 = \frac{2}{3}$

1

$-2/3$

-1

$E_1 M_2 \neq M_1$

What a mg!

Question 5.45. *If M is a martingale, and $m \leqslant n$, is $\boldsymbol{E}_m M_n = M_m$?* ← Yes (Tower + induction).

Knows $E_n M_{n+1} = M_n$ (def of mg)

$E_m M_{m+1} = M_m.$ ( )

Q: Compute $E_m M_{m+2}$:

def of mg with $n = m+1$

$$E_m M_{m+2} \overset{(\text{tower})}{=} E_m E_{m+1} M_{m+2} = E_m M_{m+1} \overset{\text{def of mg with } n=m}{=} M_m$$

By induction $E_m M_n = M_m$ $\forall m \leq n.$

Question 5.46. $\boxed{\textit{If } M \textit{ is a martingale does } \boldsymbol{E}M_n \textit{ change with } n\textit{?}}$

Guess: Does not change with n.

lemma.

Lemma: For any RV Z, $\forall n$.

$$\boxed{\boldsymbol{E} Z = \boldsymbol{E}\, \underbrace{\boldsymbol{E}_n Z}}$$

(*)

Pf: Know $\forall A \in \mathcal{G}_n$, $\sum_{\omega \in A} \boldsymbol{E}_n Z(\omega)\, \boldsymbol{p}(\omega) = \sum_{\omega \in A} Z(\omega)\, \boldsymbol{p}(\omega)$.

Know $\Omega \in \mathcal{G}_n$ $\forall n$.

Choose $A = \Omega$: $\Rightarrow \boldsymbol{E}\boldsymbol{E}_n Z = \sum_{\omega \in \Omega} \boldsymbol{E}_n Z(\omega)\, \boldsymbol{p}(\omega)$

$$(*) = \sum_{\omega \in \Omega} Z(\omega)\, \mathbb{P}(\omega) = \mathbb{E} Z.$$

Q.E.D.

Go Back to Our Question.

$$\forall n, \quad \mathbb{E} M_{n+1} \underset{\text{Lemma}}{=} \mathbb{E}\, \mathbb{E}_n M_{n+1} \underset{\text{Def of } M_n}{=} \mathbb{E} M_n.$$

$$\boxed{\Rightarrow \mathbb{E} M_{n+1} = \mathbb{E} M_n}$$

Q.E.D.

Question 5.47. *Conversely, if $\boldsymbol{E}M_n$ is constant, is M a martingale?*

Claim No.

$$EM_2 = EM_1 \not\Rightarrow E_1 M_2 = M_1$$

$$\Leftarrow \text{ true}$$

M_1

Q: $X_1, X_2, \ldots$ are all ind

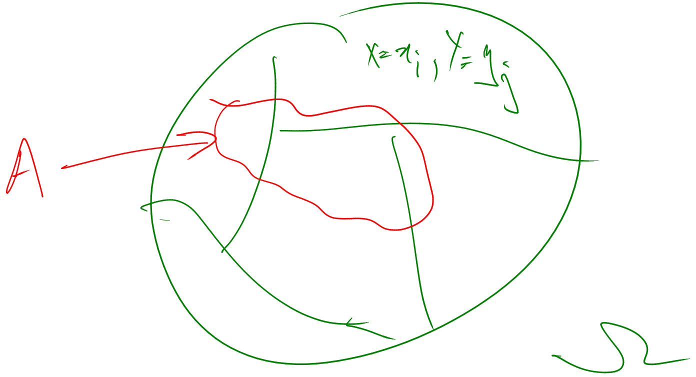
Is the proc X a mg? $\boxed{\text{Not a Mg}}$

$$E_n X_{n+1} \underset{\text{indep}}{=} \underbrace{E X_{n+1}}_{\neq X_n.} \leftarrow \text{const.}$$

n

$$\bigcup_{i,j} \{X = x_i, Y = y_j\} \leftarrow \text{covers all of } \Omega.$$

$$\bigcup_{i,j} \left[\{X = x_i, Y = y_j\} \cap A\right] \leftarrow \text{covers all of } A$$



Lecture 10 (9/22). Please enable your video if you can.

Last time: Martingales $\longrightarrow$ Fair game: $\boxed{E_n M_{n+1} = M_n.}$

(All processes are adapted)

Last time: M is a mg $\Rightarrow$ $EM_n = EM_1$ $\forall n.$

(constant expectation)

Converse is false: $EM_n = EM_1 = EM_{n+1} \not\Rightarrow M$ is a mg.

Question 5.50. *Let ξ_n be a martingale with $\boxed{\boldsymbol{E}\xi_1 = 0}$. Let Δ_n be an adapted process, $X_0 \in \mathbb{R}$ and define $X_{n+1} = X_n + \Delta_n \xi_{n+1}$. Is X a martingale?* (Need not be)

$\leftarrow E\xi_n = 0 \ \forall n.$

Remark 5.51. Think of ξ_n as the outcome of a fair game being played. You decide to bet on this game. Let Δ_n be your bet at time n; your return from this bet is $\Delta_n \xi_{n+1}$, and thus your cumulative return at time $n+1$ is $X_{n+1} = X_n + \Delta_n \xi_{n+1}$.

Pf: ① Adapted:

$$\underbrace{X_{n+1} = \underbrace{X_n}_{\substack{\mathcal{F}_n\text{-meas} \\ \text{(ind)}}} + \underbrace{\Delta_n}_{\substack{\mathcal{F}_n\text{-meas} \\ \text{(assumption)}}} \underbrace{\xi_{n+1}}_{\mathcal{F}_{n+1}\text{-meas}}}_{\mathcal{F}_{n+1} \text{ meas}} \implies \text{adapted.}$$

② NTS $E_n X_{n+1} = X_n$

Pf: $\mathbb{E}_n X_{n+1} = \mathbb{E}_n \left(X_n + \Delta_n \xi_{n+1} \right)$

$$= \mathbb{E}_n X_n + \mathbb{E}_n \left(\Delta_n \xi_{n+1} \right)$$

$$= X_n + \mathbb{E}_n \left(\Delta_n \xi_{n+1} \right)$$

$\left(\mathbb{E}_n X_n = X_n \text{ since } X_n \text{ is } \mathcal{F}_n\text{-meas} \right)$

$\left(\because \Delta_n \text{ is } \mathcal{F}_n \text{ meas} \right)$,

$$= X_n + \Delta_n \mathbb{E}_n \xi_{n+1}$$

$$= X_n + \Delta_n \, \xi_n$$ $\leftarrow$ Doesn't simplify further.

(Will work if we knew ξ_{n+1} is ind of $\mathcal{F}_n$.

Otherwise can't simplify further.)

Proof: Say M is a mg. Let $\xi_{n+1} = M_{n+1} - M_n$.

$\Delta_n \to$ any adapted process.

$$X_{n+1} = \underbrace{X_n} + \underline{\Delta_n} (M_{n+1} - M_n) = X_n + \Delta_n \xi_{n+1}.$$

Claim: In this case X is a mg.

Pf: $$E_n X_{n+1} = E_n \left(X_n + \Delta_n (M_{n+1} - M_n) \right)$$

$$= X_n + \Delta_n E_n (M_{n+1} - M_n)$$

$$= X_n \qquad \left(\because E_n M_{n+1} = M_n \Rightarrow E_n (M_{n+1} - M_n) = 0 \right)$$

5.5. Change of measure.

Example 5.52. Consider i.i.d. coin tosses with $\boldsymbol{P}(\omega_n = 1) = p_1$ and $\boldsymbol{P}(\omega_n = -1) = q_1 = 1 - p_1$. Let $u, d > 0$, $r > -1$. Let $S_{n+1}(\omega) = uS_n(\omega)$ if $\omega_{n+1} = 1$, and $S_{n+1}(\omega) = dS_n(\omega)$ if $\omega_{n+1} = -1$. Let $D_n = (1+r)^{-n}$ be the "discount factor".

Suppose we now invented a new "risk neutral" coin that comes up heads with probability $\tilde{p}_1$ and tails with probability $\tilde{q}_1 = 1 - \tilde{p}_1$. Let $\tilde{\boldsymbol{P}}, \tilde{\boldsymbol{E}}_n$ etc. denote the probability and conditional expectation with respect to the new "risk neutral" coin. Find $\tilde{p}_1$ so that $D_n S_n$ is a $\tilde{\boldsymbol{P}}$ martingale.

Theorem 5.53. *Consider a market where S_n above models a stock price, and r is the interest rate with $0 < d < 1 + r < u$. The coins land heads and tails with probability p_1 and q_1 respectively. If you have a derivative security that pays V_N at time N, then the arbitrage free price of this security at time $n \leqslant N$ is given by*

$$V_n = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n D_N V_N = (1+r)^{n-N} \tilde{\boldsymbol{E}}_n V_N .$$

(IOU a proof → 1 week).

Remark 5.54. Even though the stock price changes according to a coin that flips heads with probability p_1, the arbitrage free price is computed using conditional expectations using the *risk neutral probability*. So when computing $\tilde{\boldsymbol{E}}_n V_N$, we use our new invented "risk neutral" coin that flips heads with probability $\tilde{p}_1$ and tails with probability $\tilde{q}_1$.

- Let $p\colon \Omega \to [0,1]$ be a probability mass function on Ω, and $\boldsymbol{P}(A) = \sum_{\omega\in A} p(\omega)$ be the probability measure.
- Let $\tilde p\colon \Omega \to [0,1]$ be another probability mass function, and define a second probability measure $\tilde{\boldsymbol{P}}$ by $\tilde{\boldsymbol{P}}(A) = \sum_{\omega \in A} \tilde p(\omega)$.

Definition 5.55. We say $\boldsymbol{P}$ and $\tilde{\boldsymbol{P}}$ are equivalent if for every $A \in \mathcal{F}_N$, $\boldsymbol{P}(A) = 0$ if and only if $\tilde{\boldsymbol{P}}(A) = 0$.

Remark 5.56. When Ω is finite, $\boldsymbol{P}$ and $\tilde{\boldsymbol{P}}$ are equivalent if and only if we have $\boxed{p(\omega) = 0} \iff \boxed{\tilde p(\omega) = 0}$ for all $\omega \in \Omega$.

We let $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{E}}_n$ denote the expectation and conditional expectations with respect to $\tilde{\boldsymbol{P}}$ respectively.

$$\tilde E X = \sum X(\omega)\,\tilde p(\omega)$$

$$E X = \sum X(\omega)\,p(\omega)$$

Work out Example 5.52

Find $\tilde{p}_1$

Knows

$$\begin{tikzcd}[column sep=large]
 & uS_n \\
S_n \arrow[ur, "p_1"] \arrow[dr, "q_1"'] & \\
 & dS_n
\end{tikzcd}$$

(old coin)

$$\begin{tikzcd}[column sep=large]
 & uS_n \\
S_n \arrow[ur, "\tilde{p}_1"] \arrow[dr, "\tilde{q}_1"'] & \\
 & dS_n
\end{tikzcd}$$

(new coin)

<u>Goal:</u> Want $\tilde{P}_1$ so that $D_n S_n$ is a $\tilde{P}$ mg.

i.e. $\tilde{E}_n(D_{n+1} S_{n+1}) = D_n S_n$.

Let's compute $\tilde{E}_n S_{n+1}$:

Let $X_{n+1}(\omega) = \begin{cases} u & \omega_{n+1} = 1 \\ d & \omega_{n+1} = -1 \end{cases}$

v

Note $S_{n+1} = X_{n+1} S_n$ & X_{n+1} is $\mathcal{F}_{n+1}$ meas.

& X_{n+1} is ind of $\mathcal{F}_n$!

$$\Rightarrow \tilde{E}_n S_{n+1} = \tilde{E}_n \left(X_{n+1} S_n \right)$$

$$= S_n \tilde{E}_n X_{n+1} \qquad (\because S_n \text{ is } \mathcal{F}_n \text{ meas})$$

$$= S_n \left(\tilde{E} X_{n+1} \right) \qquad (\because X_{n+1} \text{ is ind of } \mathcal{F}_n)$$

$$= S_n\left(\tilde{p}_1 u + (1-\tilde{p}_1)\, d\right).$$

$$\Rightarrow \quad \tilde{E}_n S_{n+1} = \left(\tilde{p}_1 u + (1-\tilde{p}_1) d\right) \cdot S_n .$$

Want $D_n S_n$ to be a $\tilde{P}$ mg.

Want $\tilde{E}_n\left(D_{n+1} S_{n+1}\right) = D_n S_n$.

$$\Leftrightarrow \quad \tilde{E}_n\left((1+r)^{-(n+1)} S_{n+1}\right) = (1+r)^{-n} S_n.$$

$$\iff \tilde{E}_n S_{n+1} = (1+r) S_n.$$

$$\iff \left(\tilde{p}_1 u + (1-\tilde{p}_1) d\right) S_n = (1+r) S_n$$

$$\iff \boxed{\tilde{p}_1 u + (1-\tilde{p}_1) d = 1+r} \rightarrow$$

w

Lecture 11 (9/24). Please Enable Your Video If you Can

Last time: ① Change of measure (Equivalent Measures)

② Risk Neutral Pricing formula.

5.5. **Change of measure.**

← last time.

Example 5.52. Consider i.i.d. coin tosses with $\boldsymbol{P}(\omega_n = 1) = p_1$ and $\boldsymbol{P}(\omega_n = -1) = q_1 = 1 - p_1$. Let $u, d > 0$, $r > -1$. Let $S_{n+1}(\omega) = uS_n(\omega)$ if $\omega_{n+1} = 1$, and $S_{n+1}(\omega) = dS_n(\omega)$ if $\omega_{n+1} = -1$. Let $\boxed{D_n = (1+r)^{-n}}$ be the "discount factor".

Suppose we now invented a new "risk neutral" coin that comes up heads with probability $\tilde{p}_1$ and tails with probability $\tilde{q}_1 = 1 - \tilde{p}_1$. Let $\tilde{\boldsymbol{P}}, \tilde{\boldsymbol{E}}_n$ etc. denote the probability and conditional expectation with respect to the new "risk neutral" coin. Find $\tilde{p}_1$ so that $D_n S_n$ is a $\tilde{\boldsymbol{P}}$ martingale.

Theorem 5.53. *Consider a market where S_n above models a stock price, and r is the interest rate with $0 < d < 1 + r < u$. The coins land heads and tails with probability p_1 and q_1 respectively. If you have a derivative security that pays V_N at time N, then the arbitrage free price of this security at time $n \leqslant N$ is given by*

$$\boxed{V_n = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n D_N V_N = (1+r)^{n-N} \tilde{\boldsymbol{E}}_n V_N \,.}$$

(IOU Pf.)

Remark 5.54. Even though the stock price changes according to a coin that flips heads with probability p_1, the arbitrage free price is computed using conditional expectations using the *risk neutral probability*. So when computing $\tilde{\boldsymbol{E}}_n V_N$, we use our new invented "risk neutral" coin that flips heads with probability $\tilde{p}_1$ and tails with probability $\tilde{q}_1$.

Cond exp w.r.t the risk neutral measure $\tilde{P}$

(not P!!)

- Let $p\colon \Omega \to [0,1]$ be a probability mass function on Ω, and $\boldsymbol{P}(A) = \sum_{\omega \in A} p(\omega)$ be the probability measure. ✓ last time.
- Let $\tilde{p}\colon \Omega \to [0,1]$ be another probability mass function, and define a second probability measure $\tilde{\boldsymbol{P}}$ by $\tilde{\boldsymbol{P}}(A) = \sum_{\omega \in A} \tilde{p}(\omega)$.

Definition 5.55. We say $\boldsymbol{P}$ and $\tilde{\boldsymbol{P}}$ are equivalent if for every $A \in \mathcal{F}_N$, $\boldsymbol{P}(A) = 0$ if and only if $\tilde{\boldsymbol{P}}(A) = 0$.

Remark 5.56. When Ω is finite, $\boldsymbol{P}$ and $\tilde{\boldsymbol{P}}$ are equivalent if and only if we have $p(\omega) = 0 \iff \tilde{p}(\omega) = 0$ for all $\omega \in \Omega$.

We let $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{E}}_n$ denote the expectation and conditional expectations with respect to $\tilde{\boldsymbol{P}}$ respectively.

$$P(A) = \sum_{\omega \in A} p(\omega) \quad (\text{prob of } A \text{ occuring under } P)$$

$$\tilde{P}(A) = \sum_{\omega \in A} \tilde{p}(\omega) \quad (\text{ " \quad " \quad " \quad " \quad " } \tilde{P})$$

Work out Example 5.52. Find $\tilde{p}_1$ & $\tilde{q}_1$ so that $D_n S_n$ is a $\tilde{P}$ mg.

$$X_{n+1} = \begin{cases} u & \text{if } n+1^{\text{th}} \text{ coin is heads} \\ d & \text{" " " " tails} \end{cases}$$

$S_{n+1} = X_{n+1} S_n$ & X_{n+1} is ind of $\mathcal{F}_n$. (under $\tilde{P}$).

$$\tilde{E}_n S_{n+1} = \tilde{E}_n (X_{n+1} S_n) = S_n \tilde{E}_n X_{n+1} = S_n \tilde{E} X_{n+1}$$

$$= \underbrace{S_n \left(\tilde{p}_1 u + \tilde{q}_1 d \right)}$$

Want $D_n S_n$ to be a $\tilde{P}$ mg

$$\iff \tilde{E}_n \left(D_{n+1} S_{n+1} \right) \overset{\text{Want}}{=} D_n S_n.$$

$$\iff \frac{1}{(1+r)^{n+1}} \tilde{E}_n S_{n+1} \overset{\text{Want}}{=} \frac{1}{(1+r)^n} S_n \iff \boxed{\tilde{E}_n S_{n+1} \overset{\text{Want}}{=} (1+r) S_n}$$

Use $\tilde{E}_n S_{n+1} = \left(\tilde{p}_1 u + \tilde{q}_1 d \right) S_n.$

$$\Rightarrow (\tilde{p}_1 u + \tilde{q}_1 d) \cancel{S_n} = (1+r) \cancel{S_n}.$$

$$\Rightarrow \boxed{\tilde{p}_1 u + \tilde{q}_1 d = 1+r}$$

$$\Rightarrow \tilde{p}_1 u + (1-\tilde{p}_1) d = 1+r \iff \boxed{\tilde{p}_1 = \frac{1+r-d}{u-d}}$$

$\therefore$ If we chose $\tilde{p}_1 = \dfrac{1+r-d}{u-d}$ & $\tilde{q}_1 = 1-\tilde{p}_1 = \dfrac{u-(1+r)}{u-d}$

then $D_n S_n$ is a $\tilde{P}$ mg.

Example 5.57. Let Ω be the sample space corresponding to N i.i.d. fair coins (heads is 1, tails is -1). Let $a \in \mathbb{R}$ and define $X_{n+1}(\omega) = X_n(\omega) + \omega_{n+1} + a$. For what a is there an equivalent measure $\tilde{\boldsymbol{P}}$ such that X is a martingale?

(Try & work this out)

6. The multi-period binomial model

6.1. Risk Neutral Pricing.

- In the multi-period binomial model we assume $\Omega = \{\pm 1\}^N$ corresponds to a probability space with N i.i.d. coins.
- Let $u, d > 0$, $S_0 > 0$, and define $S_{n+1} = \begin{cases} uS_n & \omega_{n+1} = 1\,, \\ dS_n & \omega_{n+1} = -1\,. \end{cases}$
- u and d are called the up and down factors respectively.
- Without loss, can assume $d < u$.
- Always assume no coins are deterministic: $p_1 = \boldsymbol{P}(\omega_n = 1) > 0$ and $q_1 = 1 - p_1 = \boldsymbol{P}(\omega_n = -1) > 0$.
- We have access to a bank with interest rate $r > -1$.
- $D_n = (1+r)^{-n}$ be the discount factor (\$1 at time n is worth \$$D_n$ at time 0.)

Theorem 6.1. *There exists a (unique) equivalent measure $\tilde{\boldsymbol{P}}$ under which process $D_n S_n$ is a martingale* if and only if $\boxed{d < 1 + r < u}$. *In this case $\tilde{\boldsymbol{P}}$ is the probability measure obtained by tossing N i.i.d. coins with*

$$\tilde{\boldsymbol{P}}(\omega_n = 1) = \tilde{p}_1 = \frac{1+r-d}{u-d}\,, \qquad \tilde{\boldsymbol{P}}(\omega_n = -1) = \tilde{q}_1 = \frac{u-(1+r)}{u-d}\,.$$

Definition 6.2. An equivalent measure $\tilde{\boldsymbol{P}}$ under which $D_n S_n$ is a martingale is called the *risk neutral measure.*

Remark 6.3. If there are more than one risky assets, S^1, ..., S^k, then we require $D_n S^1_n$, ..., $D_n S^k_n$ to all be martingales under the risk neutral measure $\tilde{\boldsymbol{P}}$.

Remark 6.4. The *Risk Neutral Pricing Formula* says that any security with payoff V_N at time N has arbitrage free price $\boxed{V_n = \frac{1}{D_n}\tilde{\boldsymbol{E}}_n(D_N V_N)}$ at time n.

Pf of Thm 6.1:

① If $d < 1+r < u$, then chose $\tilde{p}_1 = \dfrac{1+r-d}{u-d} \in (0,1)$

$\tilde{q}_1 = \dfrac{u-(1+r)}{u-d} \in (0,1)$

By example above we know $\tilde{E}_n(D_{n+1} S_{n+1}) = D_n S_n$

$\Rightarrow \tilde{P}$ can be obtained by using iid tosses of a coin that has heads with pr $\tilde{p}_1$ & tails with pr $\tilde{q}_1$.

u

(2) Reverse direction:

<u>Only</u> choice of $\tilde{p}_1$ & $\tilde{q}_1$ for which $\tilde{E}_n(D_{n+1}S_{n+1}) = D_n S_n$

is given by $\tilde{p}_1 = \dfrac{1+r-d}{u-d}$ & $\tilde{q}_1 = \dfrac{u-(1+r)}{u-d}$.

If $1+r < d \Rightarrow \tilde{p}_1 < 0 \Rightarrow \tilde{P}$ is not a prob measure.

If $1+r = d \Rightarrow \tilde{p}_1 = 0$. $\tilde{P}$ is a prob meas

but $\tilde{P}$ is NOT equiv to P.

If $1+r > u \Rightarrow \tilde{q}_1 < 0 \Rightarrow \tilde{P}$ is not a prob meas,

& $1+r = u \Rightarrow \tilde{q}_1 = 0 \Rightarrow \tilde{P}$ is a prob meas

but NOT equiv to P.

QED.

Lecture 12 (9/27). Please enable your video if you can.

6. The multi-period binomial model

6.1. Risk Neutral Pricing.

- In the multi-period binomial model we assume $\Omega = \{\pm 1\}^N$ corresponds to a probability space with N i.i.d. coins.
- Let $u, d > 0$, $S_0 > 0$, and define $S_{n+1} = \begin{cases} uS_n & \omega_{n+1} = 1\,, \\ dS_n & \omega_{n+1} = -1\,. \end{cases}$
- u and d are called the up and down factors respectively.
- Without loss, can assume $d < u$.
- Always assume no coins are deterministic: $p_1 = \boldsymbol{P}(\omega_n = 1) > 0$ and $q_1 = 1 - p_1 = \boldsymbol{P}(\omega_n = -1) > 0$.
- We have access to a bank with interest rate $r > -1$.
- $D_n = (1+r)^{-n}$ be the discount factor (\$1 at time n is worth \$$D_n$ at time 0.)

Theorem 6.1. *There exists a (unique) equivalent measure $\tilde{\boldsymbol{P}}$ under which process $D_n S_n$ is a martingale* if and only if $d < 1 + r < u$. *In this case $\tilde{\boldsymbol{P}}$ is the probability measure obtained by tossing N i.i.d. coins with*

$$\tilde{\boldsymbol{P}}(\omega_n = 1) = \tilde{p}_1 = \frac{1 + r - d}{u - d}\,, \qquad \tilde{\boldsymbol{P}}(\omega_n = -1) = \tilde{q}_1 = \frac{u - (1+r)}{u - d}\,.$$

Definition 6.2. An equivalent measure $\tilde{\boldsymbol{P}}$ under which $D_n S_n$ is a martingale is called the *risk neutral measure.*

Remark 6.3. If there are more than one risky assets, $S^1, \dots, S^k$, then we require $D_n S_n^1, \dots, D_n S_n^k$ to all be martingales under the risk neutral measure $\tilde{\boldsymbol{P}}$.

Remark 6.4. The *Risk Neutral Pricing Formula* says that any security with payoff V_N at time N has arbitrage free price $V_n = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n(D_N V_N)$ at time n.

(Note AFP at time $0 = V_0 = \frac{1}{1} \tilde{E}(D_N V_N)$)

Example 6.5. Consider two markets in the Binomial model setup with the same u, d, r. In the first market the coin flip heads with probability 99%. In the second the coin flips heads with probability 90%. Are the price of call options in these two markets the same?

(assume $d = \frac{1}{u}$)

M.M

S

coins flip heads with prob. 99%

M.M

S

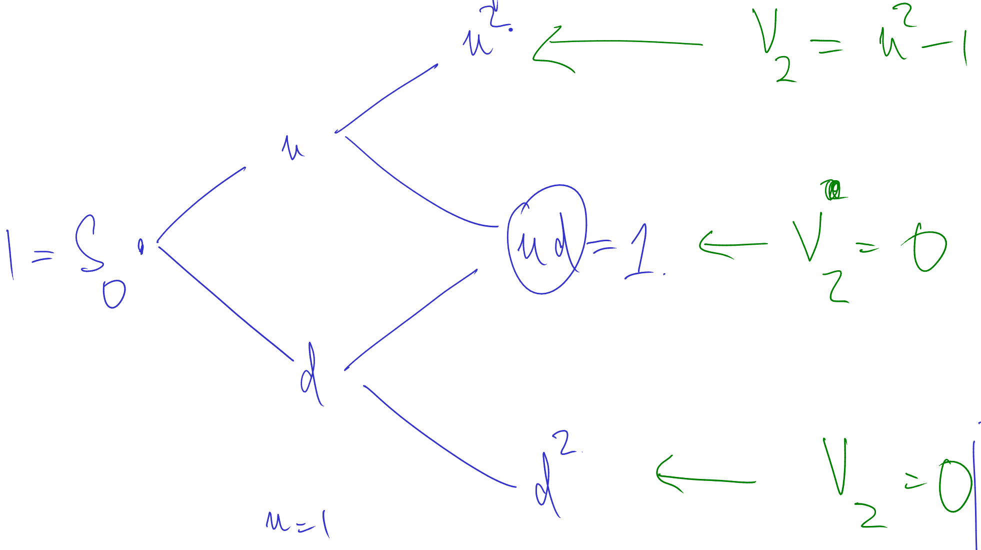
Coins flip heads with prob 90%

$N=2$. V_2 = call option payoff strike S_0

Market 1. Binomial model r, u, d

Market 2. Binomial model r, u, d

Say $S_0 = 1$ & compute V_0 (blue stock)



$$V_0 = \tilde{E}(D_2 V_2)$$

$$= \frac{(\tilde{p}_1)^2 (u^2 - 1)}{(1+r)^2}$$

$$\boxed{V_0 = \frac{1}{(1+r)^2}\left(\frac{1+r-d}{u-d}\right)^2 (u^2-1)}$$

Calculation for red stock: Exactly the same!

$$V_0 = \boxed{\frac{1}{(1+r)^2}\left(\frac{1+r-d}{u-d}\right)^2(u^2-1)}$$

- Consider an investor that starts with X_0 wealth, which he divides between cash and the stock.
- If he has Δ_0 shares of stock at time 0, then $X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$.
- We allow the investor to trade at time 1 and hold Δ_1 shares.
- Δ_1 may be random, *but must be $\mathcal{F}_1$-measurable.*
- Continuing further, we see $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$.
- *Both X and Δ are adapted processes.*

Need Δ_n to be $\mathcal{F}_n$ meas!

Definition 6.6. A *self-financing portfolio* is a portfolio whose wealth evolves according to

(self financing means no external cash flow)

$$\longrightarrow X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)\,,$$

for some adapted process Δ_n.

Theorem 6.7. *Let $d < 1 + r < u$, and $\tilde{\boldsymbol{P}}$ be the risk neutral measure, and X_n represent the wealth of a portfolio at time n. The portfolio is self-financing portfolio if and only if the discounted wealth $D_n X_n$ is a martingale under $\tilde{\boldsymbol{P}}$.*

Remark 6.8. The only thing we will use in this proof is that $D_n S_n$ is a martingale under $\tilde{\boldsymbol{P}}$. The interest rate r can be a *random* adapted process. It is also not special to the binomial model – it works for any model for which there is a risk neutral measure.

(Def of $\tilde{P}$ is that $D_n S_n$ is a $\tilde{P}$ mg)

IOU a proof of Thm 6.7.

Before proving Theorem 6.7, we consider a few consequences:

Theorem 6.9. *The multi-period binomial model is arbitrage free if and only if* $\boxed{d < 1 + r < u}$.

Remark 6.10. The first fundamental theorem of asset pricing states that a risk neutral measure exists if and only if the market is arbitrage free. (We will prove this in more generality later.)

Pf: No arbitrage: We say there is no arbitrage in the market if for any $\boxed{\text{self financing}}$ portfolio with wealth process X_n, we have: If $X_0 = 0$, & $\underbrace{X_N \geq 0}$ then $X_N = 0$ almost surely

① Clearly if $1+r \leq d$ or $1+r \geq u \longrightarrow$ there is arb.
(You check)

(2) Say $d < 1+r < u$.

NTS there is no arb in the market.

Pf: NTS If $X_0 = 0$, X_n = wealth at time n of a self financing portfolio

$\underline{X_N \geq 0}$ then $\underline{X_N = 0}$

Know by thm 6.7 that If X_n is self financing then $D_n X_n$ is a $\tilde{P}$ mg.

$$\Rightarrow \underbrace{D_0 X_0}_{=0} = \tilde{E}(D_N X_N) \qquad (\because \text{mg})$$

$$\Rightarrow \tilde{E}(\underbrace{D_N X_N}) = 0 .$$

Know $X_N \geq 0$ a.s. ($D_N > 0$).

$$\Rightarrow D_N X_N = 0 \text{ a.s.} \Rightarrow X_N = 0 \text{ a.s.}$$

Theorem 6.11 (Risk Neutral Pricing Formula)**.** *Let $d < 1 + r < u$, and V_N be an $\mathcal{F}_N$ measurable random variable. Consider a security that pays V_N at maturity time N. For any $n \leqslant N$, the arbitrage free price of this security is given by*

$$V_n = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n(D_N V_N)\,.$$

Remark 6.12. The replicating strategy can be found by backward induction. Let $\omega = (\omega', \omega_{n+1}, \omega'')$. Then

$$\Delta_n(\omega) = \frac{V_{n+1}(\omega', 1, \omega'') - V_{n+1}(\omega', -1, \omega'')}{(u-d)S_n(\omega)} = \frac{V_{n+1}(\omega', 1) - V_{n+1}(\omega', -1)}{(u-d)S_n(\omega)}$$

Proof of Theorem 6.7 part 1. Suppose X_n is the wealth of a self-financing portfolio. Need to show $D_n X_n$ is a martingale under $\tilde{\boldsymbol{P}}$.

Proof of Theorem 6.7 part 2. Suppose $D_n X_n$ is a martingale under $\tilde{\boldsymbol{P}}$. Need to show X_n is the wealth of a self-financing portfolio.

6.2. **State processes.**

Question 6.13. *Consider the N-period binomial model, and a security with payoff V_N. Let X_n be the arbitrage free price at time $n \leqslant N$, and Δ_n be the number of shares in the replicating portfolio. What is an algorithm to find X_n, Δ_n for all $n \leqslant N$? How much is the computational time?*

Theorem 6.14. *Suppose a security pays $V_N = g(S_N)$ at maturity N for some (non-random) function g. Then the arbitrage free price at time $n \leqslant N$ is given by $V_n = f_n(S_n)$, where:*

(1) $f_N(x) = V_N(x)$ *for* $x \in \operatorname{Range}(S_N)$.

(2) $f_n(x) = \dfrac{1}{1+r}(\tilde{p} f_{n+1}(ux) + \tilde{q} f_{n+1}(dx))$ *for* $x \in \operatorname{Range}(S_n)$.

Remark 6.15. Reduces the computational time from $O(2^N)$ to $O(\sum_0^N |\operatorname{Range}(S_n)|) = O(N^2)$ for the Binomial model.

Remark 6.16. Can solve this to get $f_n(x) = \dfrac{1}{(1+r)^{N-n}} \displaystyle\sum_{k=0}^{N-n} \binom{N-n}{k} \tilde{p}^k \tilde{q}^{N-n-k} f_N(x u^k d^{N-n-k})$

Question 6.17. *How do we handle other securities? E.g. Asian options (of the form $g(\sum_0^N S_k)$)?*

Lecture 13 (10/1). Please enable video if you can.

- Consider an investor that starts with X_0 wealth, which he divides between cash and the stock.
- If he has Δ_0 shares of stock at time 0, then $X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$.
- We allow the investor to trade at time 1 and hold Δ_1 shares.
- Δ_1 may be random, *but must be $\mathcal{F}_1$-measurable.*
- Continuing further, we see $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$.
- *Both X and Δ are adapted processes.*

RNM is an equivalent measure under which $D_n S_n$ is a $\tilde{P}$ mg

Definition 6.6. A *self-financing portfolio* is a portfolio whose wealth evolves according to

$$\boxed{X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)\,,}$$

for some $\boxed{\text{adapted}}$ process Δ_n.

Theorem 6.7. *Let $d < 1 + r < u$, and $\tilde{\boldsymbol{P}}$ be the risk neutral measure, and X_n represent the wealth of a portfolio at time n. The portfolio is self-financing portfolio if and only if the discounted wealth $D_n X_n$ is a martingale under $\tilde{\boldsymbol{P}}$.*

Remark 6.8. The only thing we will use in this proof is that $D_n S_n$ is a martingale under $\tilde{\boldsymbol{P}}$. The interest rate r can be a *random* adapted process. It is also not special to the binomial model – it works for any model for which there is a risk neutral measure.

IOU a proof of this theorem.

Before proving Theorem 6.7, we consider a few consequences:

Theorem 6.9. *The multi-period binomial model is arbitrage free if and only if* $d < 1 + r < u$.

Definition 6.10. We say the market is arbitrage free if for any self financing portfolio with wealth process X, we have: $X_0 = 0$ and $X_N \geqslant 0$ implies $X_N = 0$ almost surely.

Remark 6.11. The first fundamental theorem of asset pricing states that a risk neutral measure exists if and only if the market is arbitrage free. (We will prove this in more generality later.)

Pf: (1) $1+r \leq d$: Have arb (borrow cash & buy stock $\to$ arbitrage)

(2) $1+r \geq u$: Have arb (short stock & bank cash)

(3) $d < 1+r < u$. NTS $\nexists$ arb.

Know a RNM Exists. Call it $\tilde{P}$.

Say $X_0 = 0$. X = wealth process of a self fin portfolio.

$\boxed{\text{Say } X_N \geq 0}$ NTS: $X_N = 0$ almost surely.

Pf: Know (Thm 6.7) $D_n X_n$ is a $\tilde{P}$ mg.

$$\Rightarrow D_0 X_0 = \tilde{E}(D_N X_N) \qquad (\because D_n X_n \text{ mg})$$

$$\Rightarrow \tilde{E}(D_N X_N) = 0$$

$\Rightarrow$ Know
1. $\tilde{E}(D_N X_N) = 0$
2. $D_N X_N \geq 0$

$\rightarrow$ Can only happen if $D_N X_N = 0$ ($\tilde{P}$ a.s.)

Since $D_N = (1+r)^{-N} \Rightarrow X_N = 0 \quad (\tilde{P} \text{ a.s.})$

$\Rightarrow X_N = 0 \quad (P \text{ a.s.})$.

$\Rightarrow$ No arb QED.

Theorem 6.12 (Risk Neutral Pricing Formula)**.** *Let $\boxed{d < 1 + r < u}$, and V_N be an $\mathcal{F}_N$ measurable random variable. Consider a security that pays V_N at maturity time N. For any $n \leqslant N$, the arbitrage free price of this security is given by*

$$V_n = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n(D_N V_N) . \quad = (1+r)^{n-N} \tilde{E} V_N .$$

($V_N \to$ payoff of a security, e.g $V_N = (S_N - K)^+$)

Pf: Replication $\to$ Will find a self financing portfolio with wealth process X such that $X_N = V_N$.

$\Rightarrow$ AFP of security at time n $=$ X_n.

Find X_n: let $X_N = V_N$.

For $n \le N$, let $\underline{X_n} = \overline{\frac{1}{D_n} \tilde{E}_n(D_N V_N) = \frac{1}{D_n} \tilde{E}_n(D_N X_N)}$

NTS: X_n = wealth of a self fin pf

Thm 6.7, Enough to show $D_n X_n$ is a $\tilde{P}$ mg.

Pf: $\tilde{E}_n(D_{n+1} X_{n+1}) = \tilde{E}_n\left(D_{n+1} \cdot \frac{1}{D_{n+1}} \tilde{E}_{n+1}(D_N X_N)\right)$

$= \tilde{E}_n\left(\tilde{E}_{n+1}(D_N X_N)\right) \underset{\text{tower}}{=} \tilde{E}_n(D_N X_N)$

$= D_n X_n$ (def of X_n)

$\Rightarrow D_n X_n$ is a $\tilde{P}$ mg

$\Rightarrow X_n$ = wealth of a self fin Pf (Thm 6.7)

Since $X_N = V_N \Rightarrow X_n$ = AFP of sec at time n.

$\Rightarrow$ AFP of sec $= X_n = \dfrac{1}{D_n} \tilde{E}_n (D_N X_N)$. QED.

Remark 6.13. The replicating strategy can be found by backward induction. Let $\omega = (\omega', \omega_{n+1}, \omega'')$. Then

$$\Delta_n(\omega) = \frac{V_{n+1}(\omega', 1, \omega'') - V_{n+1}(\omega', -1, \omega'')}{(u-d)S_n(\omega)} = \frac{V_{n+1}(\omega', 1) - V_{n+1}(\omega', -1)}{(u-d)S_n(\omega)}$$

$\omega' = (\omega_1, \cdots \omega_n)$

$\omega'' = (\omega_{n+2}, \cdots \omega_N)$

Pf: $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$ (Wealth of our rep pf)

$X_n = V_n$ (Rep pf).

① $\omega_{n+1} = +1$: $V_{n+1}(\omega', +1, *) = \Delta_n(\omega')\, u\, S_n(\omega') + (1+r)(V_n(\omega') - \Delta_n(\omega') S_n(\omega'))$

② $\omega_{n+1} = -1$: $V_{n+1}(\omega', -1, *) = \Delta_n(\omega')\, d\, S_n(\omega') + (1+r)[V_n(\omega') - \Delta_n(\omega') S_n(\omega')]$

u

$$① - ② \Rightarrow V_{n+1}(\omega', +1) - V_{n+1}(\omega', -1) = \Delta_n(\omega')\, S_n(\omega')\,(u-d)$$

$$\Rightarrow \Delta_n(\omega') = \frac{V_{n+1}(\omega', +) - V_{n+1}(\omega', -1)}{S_n(\omega')\,(u-d)}$$

QED

Lec 14 (10/4). Please enable your video if you can

Goal: Prove Thm 6.7.

Binomial model d, r, u,

$$S_{n+1} = \begin{cases} u\, S_n & n+1^{\text{th}} \text{ coin heads,} \\ d\, S_n & \quad " \quad " \quad \text{tails.} \end{cases}$$

$\boxed{0 < d < 1+r < u.}$

Know $\exists$ a equiv measure $\tilde{P}$ $\ni$ $D_n S_n$ is a mg under $\tilde{P}$.

$\left(D_n = \dfrac{1}{(1+r)^n} \text{ discount factor} \right)$

$\boxed{\text{Thm 6.7}}$ Let X = wealth process of any investor.

X is the wealth of a $\boxed{\text{self financing}}$ portfolio

$\iff$ $D_n X_n$ is a $\tilde{P}$ mg "

Recall: **Self fin**: (No external cash flow & no looking in future)

$\hookrightarrow$ X_n is adapted

& $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$

& need Δ_n to be adapted

trading strat.

n

Intuition :

Casino

↙

Fair game = $\underline{\underline{\text{Mg}}}$.

Stock market, 1 stock

M.M interest rate r.

→ Get 1 secure asset (M.M.)

$\boxed{\text{Pays interest.}}$

→ $\boxed{\text{Discount cash}}$ & pretend $r = 0$

$\left[\text{Exp yield from disc Stock} = \frac{u p + q d}{1+r} \right]$

→ But : under $\tilde{P}$: Exp yield from disc Stock : $\underline{\frac{u\tilde{p} + d\tilde{q}}{1+r} = 1}$

Proof of Theorem 6.7 part 1. $\boxed{\text{Suppose } X_n \text{ is the wealth of a self-financing portfolio.}}$ Need to show $\boxed{D_n X_n \text{ is a martingale}}$ under $\tilde{\boldsymbol{P}}$.

assume

NTS.

Pf: Know $\exists$ an adapted trading strat $\ni$ $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$

(def of self fin).

Compute $\tilde{E}_n(D_{n+1} X_{n+1})$ (Want $\underbrace{D_n X_n}$)

$D_n = \dfrac{1}{(1+r)^n}$.

$D_{n+1}(1+r) = D_n$.

$$\tilde{E}_n(D_{n+1} X_{n+1}) = \tilde{E}_n\Big(\underbrace{\Delta_n}_{\mathcal{F}_n\text{-meas}} D_{n+1} S_{n+1} + \underbrace{D_{n+1}(1+r)(X_n - \Delta_n S_n)}_{\mathcal{F}_n\text{-meas}} \Big)$$

$$= \Delta_n \tilde{E}_n\left(D_{n+1} S_{n+1}\right) + D_n\left(X_n - \Delta_n S_n\right)$$

$$= \Delta_n \cancel{D_n} S_n + D_n X_n - \cancel{D_n \Delta_n} S_n \qquad \left(\because D_n S_n \text{ is a } \tilde{P} \text{ mg}\right).$$

$$= D_n X_n$$

QED.

Proof of Theorem 6.7 part 2. Suppose $D_n X_n$ is a martingale under $\tilde{\boldsymbol{P}}$. Need to show X_n is the wealth of a self-financing portfolio.

Assume $D_n X_n$ is $\tilde{P}$ mg.

NTF Δ_n such that $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$

& Δ_n is $\mathcal{F}_n$- meas.

Look at $\omega = (\underbrace{\omega'}_{\text{First } n}, \omega_{n+1}, \underbrace{\omega''}_{\text{last few}})$

$\omega' = (\omega_1, \dots \omega_n)$

$\omega'' = (\omega_{n+2}, \dots \omega_N)$.

$X_{n+1}(\omega) = X_{n+1}(\omega', \omega_{n+1}, \cancel{x})$

Look at $\begin{pmatrix} X_{n+1}(\omega', 1) \\ X_{n+1}(\omega', -1) \end{pmatrix} \in \mathbb{R}^2.$

Note $\left\{ \underbrace{\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \right\}$ are linearly Ind in $\mathbb{R}^2$.

Write X_{n+1} as a lin comb of

Know $\exists\ \Delta_n(\omega')$ & $\Gamma_n(\omega')$ such that

$$\begin{pmatrix} X_{n+1}(\omega', 1) \\ X_{n+1}(\omega', -1) \end{pmatrix} = \Delta_n(\omega') \underbrace{S_n(\omega') \begin{pmatrix} u \\ d \end{pmatrix}} + \Gamma_n(\omega') \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$S_n(\omega') \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} S_{n+1}(\omega', 1) \\ S_{n+1}(\omega', -1) \end{pmatrix}$$

$$\Rightarrow X_{n+1}(\omega) = \Delta_n(\omega') S_{n+1}(\omega) + \Gamma_n(\omega')$$

$\Rightarrow X_{n+1} = \Delta_n S_{n+1} + \Gamma_n$ for some adapted Δ_n & Γ_n

NTS: $\Gamma_n = \underline{(X_n - \Delta_n S_n)(1+r)}$ *

Pf: Know $D_n X_n$ is a $\tilde{P}$ mg.

Know $D_{n+1} X_{n+1} = D_{n+1} \Delta_n S_{n+1} + D_{n+1} \Gamma_n$.

$D_{n+1} = \frac{1}{(1+r)^{n+1}}$

$$\Rightarrow D_n X_n = \tilde{E}_n(D_{n+1} X_{n+1}) = \Delta_n \underbrace{\tilde{E}_n(D_{n+1} S_{n+1})}_{D_n S_n} + D_{n+1} \Gamma_n$$

$$\Rightarrow \quad D_n X_n = \underline{\Delta_n D_n} S_n + D_{n+1} \Gamma_n . \cancel{\Delta}$$

$$\Rightarrow \quad \Gamma_n = (1+r)(X_n - \Delta_n S_n)$$ which is what we wanted

QED.

Lecture 15 (10/6). Please enable video if you can.

6.2. **State processes.**

Question 6.14. *Consider the N-period binomial model, and a security with payoff V_N. Let X_n be the arbitrage free price at time $n \leqslant N$, and Δ_n be the number of shares in the replicating portfolio. What is an algorithm to find X_n, Δ_n for all $n \leqslant N$? How much is the computational time?*

X_n = AFP at time n = Wealth of Rep Port at time n

$$= \frac{1}{D_n} \tilde{E}_n (D_N V_N) = \frac{1}{(1+r)^{N-n}} \tilde{E}_n V_N \quad \left(\text{if } D_n = \frac{1}{(1+r)^n} \right)$$

$\Rightarrow$ Say maturity N = 3 months $\to$ 90 days.

Computational cost $\approx O(\#\text{ elmts in } \Omega) = O(2^N)$

$= O(2^{90})$

$2^{10} = 1024 \approx 10^3 \Rightarrow 2^{90} = (2^{10})^9 \approx 10^{27}$

Say can compute $O(10^9)$ operations in one second.

Computational time $\approx O(10^{18})$ seconds.

Life of universe $\approx 10^{19}$ seconds!!

Theorem 6.15. *Suppose a security pays* $V_N = g(S_N)$ *at maturity* N *for some (non-random) function* g*. Then the arbitrage free price at time* $n \leqslant N$ *is given by* $V_n = f_n(S_n)$*, where:*

(1) $f_N(x) = g(x)$ *for* $x \in \mathrm{Range}(S_N)$.

(2) $f_n(x) = \dfrac{1}{1+r}(\tilde{p} f_{n+1}(ux) + \tilde{q} f_{n+1}(dx))$ *for* $x \in \mathrm{Range}(S_n)$.

Remark 6.16. Reduces the computational time from $O(2^N)$ to $O(\sum_0^N |\mathrm{Range}(S_n)|) = O(N^2)$ for the Binomial model.

Remark 6.17. Can solve this to get $f_n(x) = \dfrac{1}{(1+r)^{N-n}} \displaystyle\sum_{k=0}^{N-n} \binom{N-n}{k} \tilde{p}^k \tilde{q}^{N-n-k} f_N(x u^k d^{N-n-k})$

Note: Range $S_1 = \{uS_0, dS_0\}$ $\Rightarrow$ # Range$(S_1) = 2$.

Range$(S_2) = \{u^2 S_0, udS_0, d^2 S_0\}$ $\Rightarrow$ # Range$(S_2) = 3$

$\vdots$

Range$(S_n) = \{u^n S_0, u^{n-1} d S_0, \dots, d^n S_0\}$, # elem Range$(S_n) = n$.

① Know $f_N(x) = g(x)$ $\qquad x \in \text{Range}(S_N)$

② $f_{N-1}(x) = \frac{1}{1+r}\left(f_N(ux)\tilde{p} + f_N(dx)\tilde{q}\right)$ $\qquad x \in \text{Range}(S_{N-1})$

③ $f_{N-2}(x) = \frac{1}{1+r}\left(f_{N-1}(ux)\tilde{p} + \underbrace{f_{N-1}(dx)\tilde{q}}\right)$

$$= \frac{1}{1+r}\left(\frac{1}{1+r}\left(f_N(u^2x)\tilde{p} + f_N(u\,dx)\tilde{q}\right)\tilde{p}\right.$$

$$\left. + \frac{1}{1+r}\left(f_N(udx)\tilde{p} + f_N(d^2x)\tilde{q}\right)\tilde{q}\right)$$

$$= \frac{1}{(1+r)^2}\left(f_N(u^2x)\tilde{p}^2 + 2\tilde{p}\tilde{q}\, f_N(udx) + f_N(d^2x)\tilde{q}^2 \right)$$

Pf of Thm 6.15. Backward induction.

① Let $f_N(x) = g(x)$. $\Rightarrow$ $f_N(S_N) = g(S_N) = V_N$.

② Let $n = N-1$. NTS AFP at time n is $f_n(S_n)$

where $f_n(x) = \frac{1}{1+r}\left(f_{n+1}(ux)\tilde{p} + f_{n+1}(dx)\tilde{q} \right)$

Let X_n = AFP at time n.

Know $$X_n = \frac{1}{D_n} \tilde{E}_n \left(D_N V_N \right)$$

Since $n = N-1 \Rightarrow$ $$X_n = \frac{1}{1+r} \tilde{E}_n \left(f_{n+1}(S_{n+1}) \right)$$

$$= \frac{1}{1+r} \tilde{E}_n \left(\underbrace{f_{n+1}}_{\text{not random}} \left(\underbrace{S_n}_{\mathcal{F}_n\text{-meas}} \underbrace{Y_{n+1}}_{\mathcal{F}_n\text{-ind}} \right) \right) \quad \text{where } Y_{n+1} = \begin{cases} u & \omega_{n+1} = 1 \\ d & \omega_{n+1} = -1 \end{cases}$$

$$\overset{\text{independence lemma}}{=} \frac{1}{1+r}\left(\tilde{p}\, f_{n+1}(S_n\, u) + \tilde{q}\, f_{n+1}(S_n\, d)\right)$$

$\therefore$ AFP at time n $= X_n = \underbrace{\frac{1}{1+r}\left(\tilde{p}\, f_{n+1}(S_n u) + \tilde{q}\, f_{n+1}(S_n d)\right)}_{\text{is a fn of } S_n.}$

Let $f_n(x) = \frac{1}{1+r}\left(\tilde{p}\, f_{n+1}(x u) + \tilde{q}\, f_{n+1}(x d)\right) \Rightarrow X_n = f_n(S_n)$

QED.

Above was for $n = N-1$.

In general: Backward induction

① Suppose $X_{n+1} = \text{AFP at time } n+1 = f_{n+1}(S_{n+1})$.

② AFP at time n:

$$X_n = \frac{1}{D_n}\tilde{E}_n\left(D_N V_N\right)$$

$$= \frac{1}{D_n}\tilde{E}_n \underbrace{\tilde{E}_{n+1}\left(D_N V_N\right)}$$

$$= \frac{1}{D_n}\tilde{E}_n\left(D_{n+1} X_{n+1}\right) = \frac{1}{1+r}\tilde{E}_n f_{n+1}(S_{n+1})$$

(same reason as before) $= \frac{1}{1+r}\left(f_{n+1}(u S_n)\tilde{p} + f_{n+1}(d S_n)\tilde{q} \right)$

$$= f_n(S_n), \quad \text{where } f_n(x) = \frac{1}{1+r}\left[f_{n+1}(ux)\tilde{p} + f_{n+1}(dx)\tilde{q} \right]$$

QED

Note: Above algorithm works to price any security that is a fn of the stock price at maturity (e.g. call/put options)

Question 6.18. *How do we handle other securities? E.g. Asian options (of the form $g(\sum_0^N S_k)$)?*

Eg: Asian call option strike K & maturity N.

pays $$\left[\left(\frac{1}{N+1}\sum_{n=0}^{N} S_n\right) - K\right]^+$$

Can't directly use above alg to price Asian options

but: Can if we "expand the state process."

↳ add

Definition 6.24. We say a d-dimensional process $Y = (Y^1, \dots, Y^d)$ process is a *state process* if for any security with maturity $m \leqslant N$, and payoff of the form $V_m = f_m(Y_m)$ for some (non-random) function f_m, the arbitrage free price must also be of the form $V_n = f_n(Y_n)$ for some (non-random) function f_n.

Remark 6.25. For state processes given f_N, we find f_n by backward induction. The number of computations at time n is of order $\operatorname{Range}(Y_n)$.

Remark 6.26. The fact that S_n is Markov (under $\tilde{\boldsymbol{P}}$) implies that it is a state process.

Lecture 16 (10/8). Please enable your video if you can.

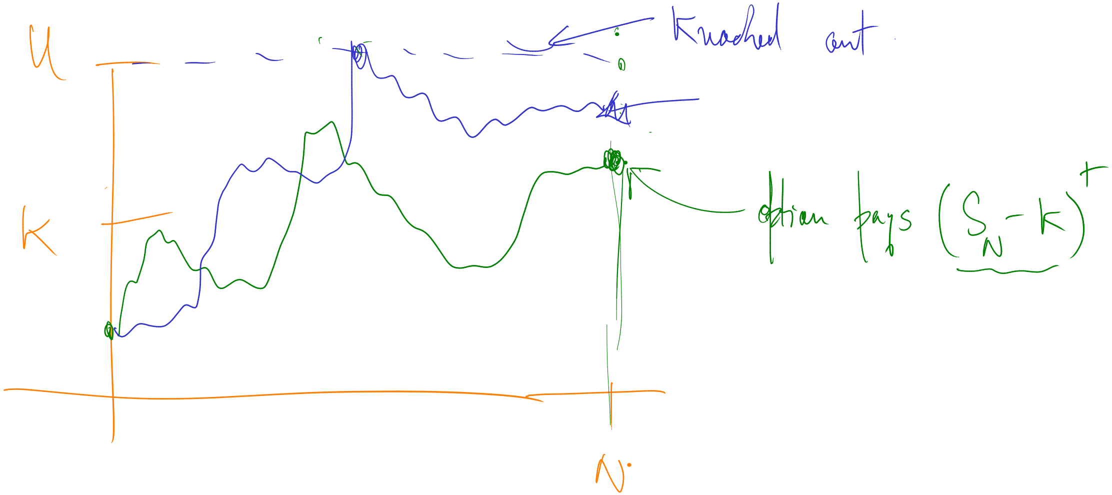
Last time: (1) AFP formula: $V_n = \frac{1}{D_n}\tilde{E}_n(D_N V_N)$ ← takes $O(2^N)$ time to compute

(2) If $\boxed{V_N = g(S_N)}$ → then can compute in time $O(N^2)$

<u>Claim:</u> $V_n = f_n(S_n)$ & $f_n(x) = \frac{1}{1+r}\left(f_{n+1}(ux)\tilde{p} + f_{n+1}(dx)\tilde{q}\right)$

$(x \in \text{Range}(S_n))$

Example 6.19 (Knockout options). An *up and out call* option with strike K and barrier U and maturity N gives the holder the option (not obligation) to buy the stock at price K at maturity time N, *provided* the stock price has never exceeded the barrier price U. If the stock price exceeds the barrier U before maturity, the option is worthless. Find an efficient algorithm to price this option.



① Payoff of option: Let $M_N = \max_{n \leq N} S_n$

$$V_N = \text{Payoff of the up \& out option} = \begin{cases} (S_N - K)^+ & M_N \leq U \\ 0 & M_N > U \end{cases}$$

$$\boxed{V_N = \mathbb{1}_{\{M_N \leq U\}} (S_N - K)^+.}$$

(Notation: if $A \subseteq \Omega$, define $\mathbb{1}_A$ to be the random variable which takes value 1 on the event A & 0 outside A.

i.e. $$\mathbb{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A. \end{cases}$$

$$\Rightarrow \mathbb{1}_{\{M_N \le u\}} (S_N - K)^+ = \begin{cases} (S_N - K)^+ & \text{on } \{M_N \le u\}, \\ 0 & \text{on } \{M_N \le u\}^c \end{cases}$$

u

Idea: Let $M_n = \max_{k \leq n} S_k$. ($M_n$ is an adapted process)

$\boxed{\text{Hope}}$ $V_n = f_n(M_n, S_n)$ & find a recurrence rel for f_n.

$\hookrightarrow$ ① Know for $n = N$, $V_N = f_N(M_N, S_N)$

where $f_N(m, s) = \begin{cases} (s-K)^+ & m \leq U \\ 0 & m > U \end{cases}$

② Backward induction: Suppose for time $n+1$ we know

$$V_{n+1} = f_{n+1}(M_{n+1}, S_{n+1})$$

want $V_n = f_n(M_n, S_n)$ for some fn f_n.

Know $$V_n = \frac{1}{D_n} \tilde{E}_n (D_{n+1} V_{n+1}) = \frac{1}{1+r} \tilde{E}_n V_{n+1}$$

$$= \frac{1}{1+r} \tilde{E}_n \left[f_{n+1}(M_{n+1}, S_{n+1}) \right]$$

<u>Want</u> $f_n(M_n, S_n)$

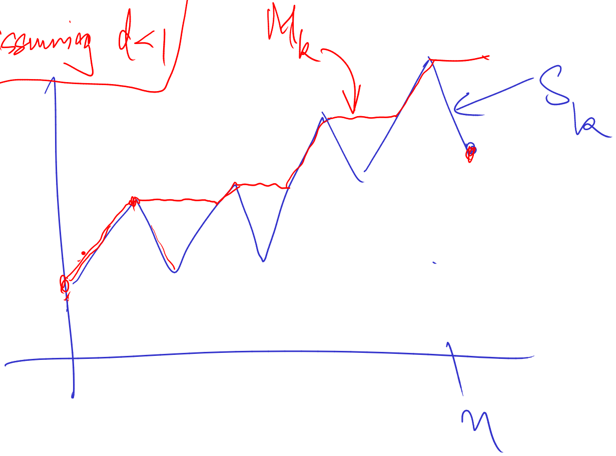
To find f_n write M_{n+1} & S_{n+1} in terms of M_n & S_n.

$$S_{n+1} = \begin{cases} uS_n & \omega_{n+1} = 1 \\ dS_n & \omega_{n+1} = -1 \end{cases} \Rightarrow S_{n+1} = X_{n+1} S_n, \text{ where } X_{n+1} = \begin{cases} u & \omega_{n+1} = 1 \\ d & \omega_{n+1} = -1 \end{cases}$$

$$M_{n+1} = \begin{cases} \max\{M_n, uS_n\} & \text{if } \omega_{n+1} = +1 \\ \end{cases}$$

$\Big(\quad M_n$

otherwise. $\boxed{\text{assuming } \alpha < 1}$



M_k

maximum

$$\Rightarrow M_{n+1} = \left(M_n \vee u S_n\right) \mathbb{1}_{\{\omega_{n+1} = 1\}} + M_n \, \mathbb{1}_{\{\omega_{n+1} = -1\}}$$

$$\Rightarrow V_n = \frac{1}{1+r} \tilde{E}_n f_{n+1}\left(M_{n+1}, S_{n+1}\right)$$

$$= \frac{1}{1+r}\,\tilde{E}_n\, f_{n+1}\Big(\underbrace{M_n v(uS_n)}_{\mathcal{F}_n\text{-meas}}\, \underbrace{\mathbb{1}_{\{\omega_{n+1}=1\}}}_{\mathcal{F}_n\text{-indep}} + \underbrace{M_n}_{\mathcal{F}_n\text{-meas}}\, \underbrace{\mathbb{1}_{\{\omega_{n+1}=-1\}}}_{\mathcal{F}_n\text{-indep}}\Big)\, \underbrace{X_{n+1}}_{\mathcal{F}_n\text{-indep}}\, \underbrace{S_n}_{\mathcal{F}_n\text{-meas}} \Big)$$

indep lemma

$\mathcal{F}_n$- meas

$\mathcal{F}_n$- indep

$$= \frac{1}{1+r}\Big(\tilde{p}\, f_{n+1}\big(M_n v(uS_n)\cdot 1 + 0,\ uS_n\big)$$

$$+ \tilde{q}\, f_{n+1}\big(M_n v(uS_n)\cdot 0 + M_n,\ dS_n\big)$$

$$V_n = \frac{1}{1+r}\left(\tilde{p}\, f_{n+1}\big(M_n v(uS_n), uS_n\big) + \tilde{q}\, f_{n+1}\big(M_n, dS_n\big)\right)$$

$$\Rightarrow V_n = f_n(\underbrace{M_n, S_n}_{\text{process } Y}) \quad \text{where} \quad \boxed{f_n(m,s) = \frac{1}{1+r}\left(\tilde{p}\, f_{n+1}\big(m\, v(us), us\big) + \tilde{q}\, f_{n+1}(m, ds)\right)}$$

Gives me a recursive relation to compute f_n in terms of f_{n+1}

Definition 6.20. We say a d-dimensional process $Y = (Y^1, \dots, Y^d)$ process is a *state process* if for any security with maturity $m \leqslant N$, and payoff of the form $V_m = f_m(Y_m)$ for some (non-random) function f_m, the arbitrage free price must also be of the form $V_n = f_n(Y_n)$ for some (non-random) function f_n.

Remark 6.21. For state processes given f_N, we typically find f_n by backward induction. The number of computations at time n is of order $\text{Range}(Y_n)$.

Remark 6.22. The fact that S_n is Markov (under $\tilde{\boldsymbol{P}}$) implies that it is a state process.

E.g. $Y_n = (S_n)$ is a state process.

Eg: $M_n = \max\limits_{k \le n} S_k$ & set $\left.\begin{array}{l} Y^1_n = M_n \\ Y^2_n = S_n \end{array}\right\} Y = (Y^1, Y^2)$

Y is a state process (as follows from Knock out eg.)

Lecture 17 (10/11). Please enable video if you can

(1) $V_n = \frac{1}{D_n} \tilde{E}_n (D_N V_N)$ $\leftarrow$ takes $O(2^N)$ time to compute.

(2) If $\boxed{V_N = g(S_N)}$,

$$f_n(x) \doteq \frac{1}{1+r} \left(\tilde{p} f_{n+1}(ux) + \tilde{q} f_{n+1}(dx) \right)$$

$$\boxed{\& \ f_N(x) = g(x).}$$

$V_n = f_n(S_n)$ $\longrightarrow$ takes $\underline{O(N^3)}$ time to compute

(3) If V_N is not in this form (Eg. Knockout options), find fast algorithms

Definition 6.20. We say a d-dimensional process $Y = (Y^1, \dots, Y^d)$ process is a *state process* if for any security with maturity $m \leqslant N$, and payoff of the form $V_m = f_m(Y_m)$ for some (non-random) function f_m, the arbitrage free price must also be of the form $V_n = f_n(Y_n)$ for some (non-random) function f_n.

Remark 6.21. For state processes given f_N, we typically find f_n by backward induction. The number of computations at time n is of order Range(Y_n).

Remark 6.22. The fact that S_n is Markov (under $\tilde{\boldsymbol{P}}$) implies that it is a state process.

Convention: superscript $\longrightarrow$ coordinate

subscript $\longrightarrow$ time.

State process: If payoff is a fn of the state then AFP is also a fn of the state

Theorem 6.23. *Let $Y = (Y^1, \ldots, Y^d)$ be a d-dimensional process. Suppose we can find functions $g_1, \ldots, g_N$ such that $Y_{n+1}(\omega) = g_{n+1}(Y_n(\omega), \omega_{n+1})$. Then Y is a state process.*

① Pf of thm: Consider a sec. that pays $\overbrace{f_N(Y_N)}^{V_N.}$ at time N.

NTS $\forall\ n \leq N$, AFP at time n is some fn of Y_n.

② Say AFP at time $n+1$ is $f_{n+1}(Y_{n+1})$. [True for $n = N-1$]

NTS AFP at time n is $f_n(Y_n)$ for some fn f_n that I will find.

Know AFP at time n = $V_n = \dfrac{1}{1+r}\tilde{E}_n \underline{V_{n+1}}$

$$\Rightarrow V_n = \frac{1}{1+r}\tilde{E}_n f_{n+1}(Y_{n+1}) = \frac{1}{1+r}\tilde{E}_n f_{n+1}\Big(g_{n+1}\big(\underbrace{Y_n}_{\mathcal{F}_n\text{-meas}}, \underbrace{\omega_{n+1}}_{\text{ind of } \mathcal{F}_n}\big)\Big)$$

$$\overset{\text{indep lemma}}{=} \frac{1}{1+r}\Big(\tilde{p}\, f_{n+1}\big(g_{n+1}(Y_n, +1)\big) + \tilde{q}\, f_{n+1}\big(g_{n+1}(Y_n, -1)\big)\Big)$$

Some fn of Y_n

$$\Rightarrow V_n = f_n(Y_n) \text{ where } \boxed{f_n(y) = \frac{1}{1+r}\Big(\tilde{p}\, f_{n+1}\big(g_{n+1}(y,+1)\big) + \tilde{q}\, f_{n+1}\big(g_{n+1}(y,-1)\big)\Big)}$$

QED

Note: Gives a recurrence rel to find f_n in terms of f_{n+1}

computations $\approx O(\text{Range}(Y_n))$

Question 6.24. *Is $Y_n = S_n$ a state process?* ← Yes (started with this)

Question 6.25. *Is $Y_n = \max_{k \leqslant n} S_k$ a state process?* ← Not a state process.

Question 6.26. *Is $Y_n = (S_n, \max_{k \leqslant n} S_n)$ a state process?*

$M_n = \max_{k \leq n} S_k$. → state process? ← NO

② $Y_n = (S_n, M_n)$ → state process? ← Yes.

↳ To check this is a state process only need to write

Y_{n+1} as some fn of Y_n & ω_{n+1}, Note $Y'_{n+1} = S_{n+1} = S_n\left(u\,\mathbb{1}_{\omega_{n+1}=1} + d\,\mathbb{1}_{\omega_{n+1}=-1}\right)$

$$= \underbrace{Y'_n\left(u\,\mathbb{1}_{\omega_{n+1}=1} + d\,\mathbb{1}_{\omega_{n+1}=-1}\right)}_{\text{some fn of } Y_n \text{ \& } \omega_{n+1}.}$$

(Recall $\mathbb{1}_A \to$ ~~w~~ RV

$$\mathbb{1}_A = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$
)

$$Y^2_{n+1} = M_{n+1} = \max_{k \le n+1} S_k = \max\left(S_{n+1}, \max_{k \le n} S_k \right)$$

$$= \underbrace{\max\left(Y^1_n \left(\mathbb{1}_{\{\omega_{n+1}=1\}} u + \mathbb{1}_{\omega_{n+1}=-1} d \right), Y^2_n \right)}_{\text{Same fn of } Y_n \text{ \& } \omega_{n+1}}$$

Question 6.27. *Let* $A_n = \sum_0^n S_k$*. Is* A_n *a state process?* ← NO

Question 6.28. *Is* $Y_n = (S_n, A_n)$ *a state process?*

Yes (related to HW)

$S_0 = 1$

$\text{Range}(S_1) = \{u, d\}$

$\text{Rge}(S_2) = \{u^2, ud, d^2\}$

$\text{Rge}(A_1) = \{1+u, 1+d\}$

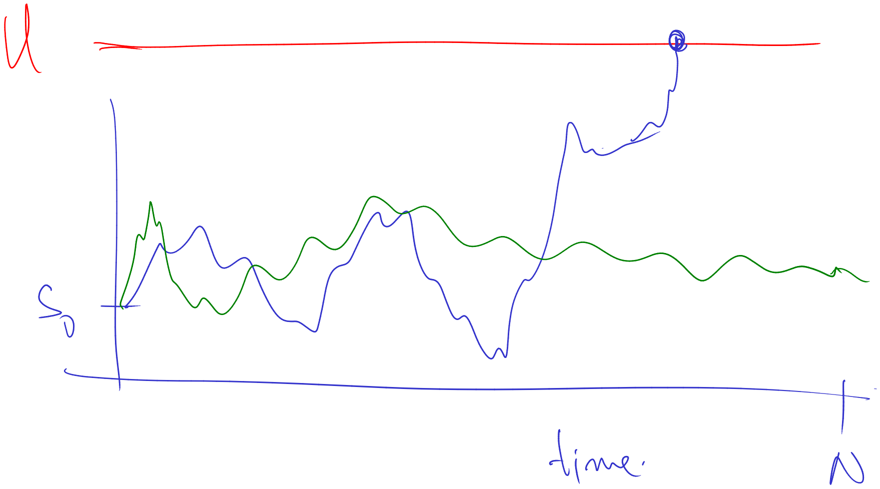
$\text{Rge}(A_2) = \{1+u+u^2, 1+u+ud, 1+d+ud, 1+d+d^2\}$

$\text{Rge}(A_3) =$ 8 vals

6.3. **Options with random maturity.** Consider the N period binomial model with $0 < d < 1 + r < u$.

Example 6.29 (Up-and-rebate option). Let $A, U > 0$. The up-and-rebate option pays the face value A at the first time the stock price exceeds U (up to maturity time N), and nothing otherwise. Explicitly, let $\tau = \min\{n \leqslant N \mid S_n \geqslant U\}$, and let $\sigma = \tau \wedge N$. The up-and-rebate options pays $A\mathbf{1}_{\tau \leqslant N}$ at the random time σ.

Remark 6.30. By convention $\min \emptyset = \infty$.



Definition 6.31. We say a random variable τ is a $\boxed{\textit{stopping time}}$ if:

(1) $\tau\colon \Omega \to \{0, \dots, N\} \cup \infty$
(2) For all $n \leqslant N$, the event $\{\tau \leqslant n\} \in \mathcal{F}_n$.

Remark 6.32. We say τ is a finite stopping time if $\tau < \infty$ almost surely.

Remark 6.33. The second condition above is equivalent to requiring $\{\tau = n\} \in \mathcal{F}_n$ for all n.

Lecture 18 (10/13). Please enable your video if you can

Notation: $a \wedge b = \min\{a, b\}$

$a \vee b = \max\{a, b\}$.

6.3. **Options with random maturity.** Consider the N period binomial model with $0 < d < 1 + r < u$.

Example 6.29 (Up-and-rebate option). Let $A, U > 0$. The up-and-rebate option pays the face value A at the first time the stock price exceeds U (up to maturity time N), and nothing otherwise. Explicitly, let $\tau = \min\{n \leqslant N \mid S_n \geqslant U\}$, and let $\sigma = \tau \wedge N$. The up-and-rebate options pays $A\mathbf{1}_{\tau \leqslant N}$ at the random time σ.

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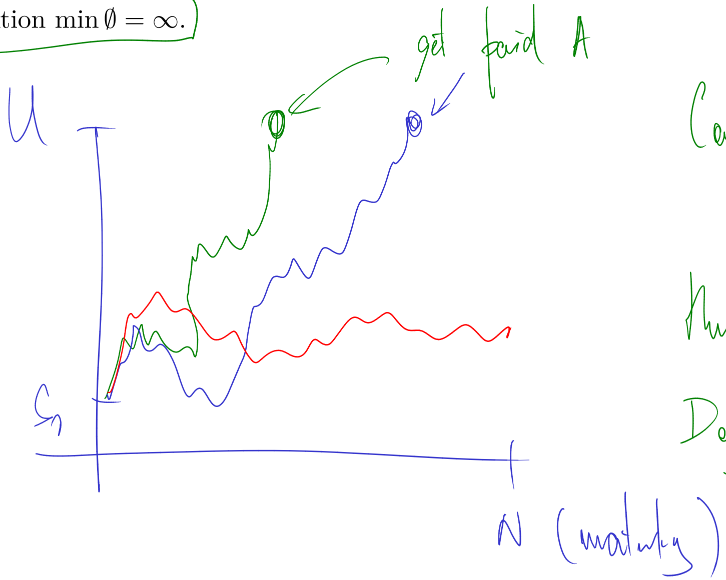


Could happen that $\{S_n \leq U$ $\forall n \leq N$.

then $\{n \leq N \mid S_n \geq U\} = \emptyset$

Define $\min \emptyset = +\infty$.

Definition 6.31. We say a random variable τ is a *stopping time* if:

(1) $\tau\colon \Omega \to \{0, \dots, N\} \cup \{\infty\}$

(2) For all $n \leqslant N$, the event $\{\tau \leqslant n\} \in \mathcal{F}_n$.

Remark 6.32. We say τ is a finite stopping time if $\tau < \infty$ almost surely.

Remark 6.33. The second condition above is equivalent to requiring $\{\tau = n\} \in \mathcal{F}_n$ for all n.

$\tau \longrightarrow$ ~~as~~ time we deside to stop playing a game

$\{\tau = n\} \longrightarrow$ event we decided to stop playing at time n.

Require $\{\tau = n\} \in \mathcal{F}_n$ (only uses first n coin tosses).

Question 6.34. *Is $\tau = 5$ a stopping time?* Yes: check: ① $\text{Range}(\tau) = \{5\} \subseteq \{0, \dots N\} \cup \{\infty\}$ ✓

Question 6.35. *Is the first time the stock price hits U a stopping time?*

Question 6.36. *Is the last time the stock price hits U a stopping time?*

τ = first time stock price exceeds U.

τ = last time " " " U.

$$\tau = \min\{n \leq N \mid S_n \geq U\}$$

$$\tau = \max\{n \leq N \mid S_n \geq U\}$$

Guess τ is NOT a stopping time Yes

② NTS $\{\tau = n\} \in \mathcal{F}_n \ \forall n$.

$$\{\tau = n\} = \begin{cases} \Omega & n = 5 \\ \emptyset & \text{OW.} \end{cases}$$

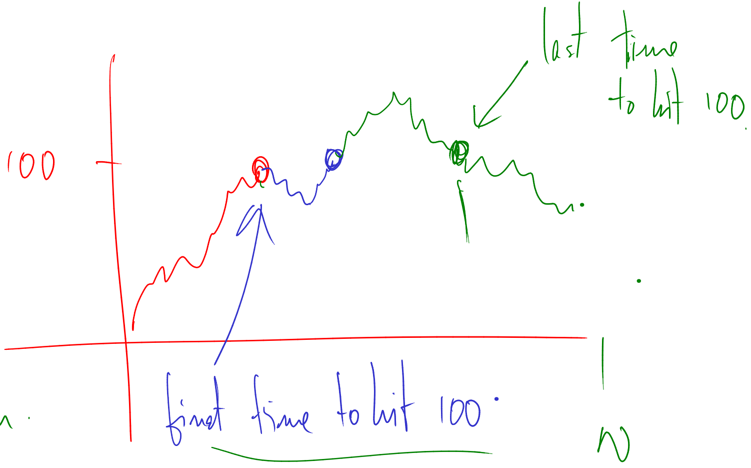
$\Rightarrow \Omega \in \mathcal{F}_n \ \forall n$ & $\emptyset \in \mathcal{F}_n \ \forall n$

$\Rightarrow \{\tau = n\} \in \mathcal{F}_n \ \forall n$ ✓

Guess τ is a stopping time. Yes.

To see τ is a stopping time $\in \mathcal{F}_n$.

$$\{\tau = n\} = \{S_k < U \ \forall k < n \ \& \ \underbrace{S_n \geq U}_{\in \mathcal{F}_n}\} \in \mathcal{F}_n.$$



$$\{\tau = n\} = \{\underbrace{S_k < U \ \forall k > n}_{\text{Requires info on coin tosses after } n \ \notin \mathcal{F}_n!} \ \& \ \underbrace{S_n \geq U}_{\in \mathcal{F}_n}\}$$

need not $\notin \mathcal{F}_n$.

Question 6.37. *If σ and τ are stopping times, is $\sigma \wedge \tau$ a stopping time? How about $\sigma \vee \tau$?*

$\max\{\sigma, \tau\}$

Think about both these

$\min\{\sigma, \tau\}$

- Let G be an adapted process, and σ be a *finite* stopping time.
- Consider a derivative security that pays G_σ at the random time σ.
- Note $G_\sigma = \sum_{n=0}^{N} G_n \mathbf{1}_{\{\sigma=n\}}$
- Let $(X_0, (\Delta_n))$ be a self-financing portfolio, and X_n at time n be the wealth of this portfolio at time n.

(i.e σ is a stopping time & $P\{\sigma < \infty\} = 1$.)

($G_\sigma = G_n$ when $\sigma = n$).

Definition 6.38. A self-financing portfolio with wealth process X is a replicating strategy if $X_\sigma = G_\sigma$.

Theorem 6.39. *The security with payoff G_σ (at the stopping time σ) can be replicated. The arbitrage free price is given by*

$$X_n \mathbf{1}_{\{\sigma \geqslant n\}} = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n (D_\sigma G_\sigma \mathbf{1}_{\{\sigma \geqslant n\}})$$

Remark 6.40. The only thing required for the proof of Theorem 6.39 is the fact that X_n is the wealth of a self-financing portfolio if and only if $D_n X_n$ is a $\tilde{\boldsymbol{P}}$ martingale.

Proof: Let $Z = D_\sigma G_\sigma$ (some R.V.)

Let $X_n = \frac{1}{D_n} \tilde{E}_n(Z) = \frac{1}{D_n} \tilde{E}_n(D_\sigma G_\sigma)$

- $\boxed{\text{Claim 1} \quad X_n \text{ is the wealth of a self financing portfolio.}}$

- Claim 2 $\quad X_T = G_T$.

(Note: Claim 1 + 2 $\Rightarrow$ X is a replicating portfolio of the security with payoff G_T at time T.

$\Rightarrow$ AFP of security at time $n \leq T$ is X_n.

Note $\quad \mathbb{1}_{\{n=T\}} X_n = \mathbb{1}_{\{n=T\}} \frac{1}{D_n} \tilde{E}_n (D_T G_T)$

(Note $\{\tau = n\} \in \mathcal{F}_n$ since τ is a stopping time $\Rightarrow \mathbb{1}_{\{\tau = n\}}$ is $\mathcal{F}_n$ meas)

$$= \frac{1}{D_n} \tilde{E}_n \left(\mathbb{1}_{\{n=\tau\}} \underline{D_\tau G_\tau} \right)$$

$$= \frac{1}{D_n} \tilde{E}_n \left(\mathbb{1}_{\{n=\tau\}} \underline{D_n G_n} \right)$$

$$\therefore \mathbb{1}_{\{n=\tau\}} X_n = \frac{1}{D_n} \tilde{E}_n \left(\mathbb{1}_{\{n=\tau\}} D_n G_n \right)$$

$$= \mathbb{1}_{\{n=\tau\}} \underline{G_n}$$

$\Rightarrow X_\tau = G_\tau$

($\Rightarrow$ Claim 2)

Pf of claim 1: NTS X_n = wealth of self fin port

Enough to show $D_n X_n$ is a $\tilde{P}$ mg

But $D_n X_n = \tilde{E}_n Z = \tilde{E}_n \left(\overbrace{D_T G_T} \right)$

$$\tilde{E}_n \left(D_{n+1} X_{n+1} \right) = \tilde{E}_n \left(\tilde{E}_{n+1} \left(D_T G_T \right) \right) \overset{\text{tower}}{=} \tilde{E}_n \left(D_T G_T \right) = D_n X_n$$

QED.

Lecture 19 (10/15). Please enable your video if you can.

$a(bc) = (ab)c$ (but not on a computer!)

$\hookrightarrow$ error $\approx 10^{-17}$

range: $(S_{n+1}) \xrightarrow{\text{on computer}} \{us \mid s \in \mathrm{Range}(S_n)\} \cup \{ds \mid s \in \mathrm{Range}(S_n)\}$. lots of repeats

Smaller range $\longrightarrow$ less computations.

Last time:

~~[illegible]~~ Stopping time: Play a game. Stop at time τ. (random)

Need $\{\tau = n\} \in \mathcal{F}_n$ (only dep on 1^{st} n coin tosses)

stop at time n

Def: τ is a stopping time if ① $\tau : \Omega \longrightarrow \{0, 1, \dots N\} \cup \{\infty\}$

& ② Need $\{\tau = n\} \in \mathcal{F}_n$ $\forall n$.

(Note: ② $\iff$ ②': Need $\{\tau \le n\} \in \mathcal{F}_n$ $\forall n$).

- Let G be an adapted process, and σ be a *finite* stopping time.
- Note $G_\sigma = \sum_{n=0}^{N} G_n \mathbf{1}_{\sigma=n}$.
- Let $(X_0, (\Delta_n))$ be a self-financing portfolio, and X_n at time n be the wealth of this portfolio at time n.

Definition 6.38. Consider a derivative security that pays G_σ at the random time σ. A self-financing portfolio with wealth process X is a replicating strategy if $X_\sigma = G_\sigma$.

Remark 6.39. If a replicating strategy exists, then at any time before σ, the wealth of the replicating strategy must equal the arbitrage free price V. That is, $\mathbf{1}_{\{n \leqslant \sigma\}} X_n = \mathbf{1}_{\{n \leqslant \sigma\}} V_n$.

Binomial model & $0 < d < 1 + r < u$

Theorem 6.40. *The security with payoff G_σ (at the stopping time σ) can be replicated. The arbitrage free price is given by*

$$\boxed{V_n \mathbf{1}_{\{\sigma \geqslant n\}} = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n (D_\sigma G_\sigma \mathbf{1}_{\{\sigma \geqslant n\}})}$$

Remark 6.41. The only thing required for the proof of Theorem 6.40 is the fact that X_n is the wealth of a self-financing portfolio if and only if $D_n X_n$ is a $\tilde{\boldsymbol{P}}$ martingale.

Note $\{\sigma \geq n\} \in \mathcal{F}_n \Rightarrow \mathbb{1}_{\{\sigma \geq n\}}$ is $\mathcal{F}_n$ - meas

$$\Rightarrow \tilde{E}_n\left(\mathbb{1}_{\{\sigma \geq n\}} D_\sigma G_\sigma\right) = \mathbb{1}_{\{\sigma \geq n\}} \tilde{E}_n(D_\sigma G_\sigma)$$

Note $\{\tau = n\} \in \mathcal{F}_n$. $\quad$ ($\because$ τ is a stopping time)

$\{\tau \le n\} \in \mathcal{F}_n$ $\quad$ (Yes: $\{\tau \le n\} = \bigcup_{k=0}^{n} \underbrace{\{\tau = k\}}_{\in \mathcal{F}_k} \in \mathcal{F}_n$.)

$$\{\tau > n\} = \underbrace{\{\tau \le n\}^c}_{\in \mathcal{F}_n}$$

Also $\{\tau > n\} \in \mathcal{F}_n$

Pf of Thm 6.40: Let $X_n = \frac{1}{D_n} \tilde{E}_n (D_\sigma G_\sigma)$

Claim ① X = wealth of a self fin portfolio
Claim ② $X_\sigma = G_\sigma$

$\Rightarrow X$ is a rep portfolio.

Once we know X = wealth of rep port,

$\mathcal{F}_n$-meas

Then $V_n \mathbb{1}_{\{n \leq \sigma\}} = X_n \mathbb{1}_{\{n \leq \sigma\}} = \mathbb{1}_{\{n \leq \sigma\}} \frac{1}{D_n} \tilde{E}_n (D_\sigma G_\sigma)$

$= \frac{1}{D_n} \tilde{E}_n \left(D_\sigma G_\sigma \mathbb{1}_{\{n \leq \sigma\}} \right)$

= formula we wanted to prove.

Pf of claim 1: X is wealth of a self fin port

$\iff D_n X_n$ is a $\tilde{P}$ mg

Only NTS $D_n X_n$ is a $\tilde{P}$ mg.

Compute $$\tilde{E}_n\left(D_{n+1} X_{n+1}\right) = \tilde{E}_n \cancel{D_{n+1}} \overbrace{\tilde{E}_{n+1}\left(D_\sigma G_\sigma\right) \cdot \frac{1}{\cancel{D_{n+1}}}}^{\text{formula for } X_{n+1}}$$

$$\overset{\text{tower}}{=} \tilde{E}_n\left(D_\sigma G_\sigma\right) = D_n X_n \quad (\text{formula for } X_n).$$

QED (Claim 1).

Pf of claim 2: NTS $X_\tau = G_\tau$.

Enough to show $\forall n$, $\mathbb{1}_{\{\tau=n\}} X_\tau = \mathbb{1}_{\{\tau=n\}} G_\tau$.

Let's prove this: $LHS = \mathbb{1}_{\{\tau=n\}} X_{\underline{\underline{\tau}}} = \mathbb{1}_{\{\tau=n\}} X_{\underline{\underline{n}}}$

$$= \mathbb{1}_{\{\tau=n\}} \frac{1}{D_n} \tilde{E}_n (D_\tau G_\tau)$$

$$\overbrace{}^{\mathcal{F}_n\text{- meas}}$$

$$= \frac{1}{D_n} \tilde{E}_n \left(\mathbb{1}_{\{\tau = n\}} D_\tau G_\tau \right) = \frac{1}{D_n} \tilde{E}_n \left(\underbrace{\mathbb{1}_{\{\sigma = n\}} D_n G_n}_{\mathcal{F}_n \text{ meas}} \right)$$

$$= \frac{1}{\cancel{D_n}} \mathbb{1}_{\{\sigma = n\}} \cancel{D_n} G_n = \mathbb{1}_{\{\sigma = n\}} G_n$$

QED.

Proposition 6.42. *The wealth of the replicating portfolio (at times before σ) is uniquely determined by the recurrence relations:*

$$X_N \mathbf{1}_{\{\sigma=N\}} = G_N \mathbf{1}_{\{\sigma=N\}}$$

$$X_n \mathbf{1}_{\{\sigma \geqslant n\}} = G_n \mathbf{1}_{\{\sigma=n\}} + \frac{1}{1+r} \mathbf{1}_{\{\sigma>n\}} \tilde{\boldsymbol{E}}_n X_{n+1} .$$

If we write $\omega = (\omega', \omega_{n+1}, \omega'')$ with $\omega' = (\omega_1, \dots, \omega_n)$, then we know in the Binomial model we have

$$\tilde{\boldsymbol{E}}_n X_{n+1}(\omega) = \tilde{\boldsymbol{E}}_n X_{n+1}(\omega') = \tilde{p} X_{n+1}(\omega', 1) + \tilde{q} X_{n+1}(\omega', -1) .$$

As before, we will use state processes to find practical algorithms to price securities.

Example 6.43. Let $A, U > 0$. The up-and-rebate option pays the face value A at the first time the stock price exceeds U (up to maturity time N), and nothing otherwise. Find an efficient way to compute the arbitrage free price of this option.

Proposition 6.44. *Let $Y = (Y^1, \dots, Y^d)$ be a d-dimensional process such that for every n we have $Y_{n+1}(\omega) = h_{n+1}(Y_n(\omega), \omega_{n+1})$ for some deterministic function h_{n+1}. Let A_1, …, $A_N \subseteq \mathbb{R}^d$, with A_N $\mathbb{R}^d$, and define the stopping time σ by*

$$\sigma = \min\{n \in \{0, \dots, N\} \mid Y_n \in A_n\}\,.$$

Let g_0, …g_N be N deterministic functions on $\mathbb{R}^d$, and consider a security that pays $G_\sigma = g_\sigma(Y_\sigma)$. The arbitrage free price of this security is of the form $V_n \mathbf{1}_{\{\sigma \geqslant n\}} = f_n(Y_n)\mathbf{1}_{\{\sigma \geqslant n\}}$. The functions f_n satisfy the recurrence relation

$$f_N(y) = g_N(y)$$

$$f_n(y) = \mathbf{1}_{\{y \in A_n\}} g_n(y) + \frac{\mathbf{1}_{\{y \notin A_n\}}}{1+r}\Big(\tilde{p} f_{n+1}(h_{n+1}(y, 1)) + \tilde{q} f_{n+1}(h_{n+1}(y, -1))\Big)$$

6.4. **Optional Sampling.** Consider a market with a few risky assets and a bank.

Question 6.45. *If there is no arbitrage opportunity at time N, can there be arbitrage opportunities at time $n \leqslant N$? How about at finite stopping times?*

Proposition 6.46. *There is no arbitrage opportunity at time N if and only if there is no arbitrage opportunity at any finite stopping time.*

Question 6.47. *Say M is a martingale. We know $\boldsymbol{E}M_n = \boldsymbol{E}M_0$ for all n. Is this also true for stopping times?*

Theorem 6.48 (Doob's optional sampling theorem)**.** *Let τ be a bounded stopping time and M be a martingale. Then $\boldsymbol{E}_n M_\tau = M_{\tau \wedge n}$.*

Proposition 6.49. *Suppose a market admits a risk neutral measure. If X is the wealth of a self-financing portfolio and τ is a finite stopping time such that $X_0 = 0$, and $X_\tau \geqslant 0$, then $X_\tau = 0$.*

Remark 6.50. This is simply an alternate proof of Proposition 6.46.

Question 6.51 (Gamblers ruin)**.** *Suppose* $N = \infty$*. Let* X_n *be i.i.d. random variables with mean* 0*, and let* $S_n = \sum_1^n X_k$*. Let* $\tau = \min\{n \mid S_n = 1\}$*. (It is known that* $\tau < \infty$ *almost surely.) What is* $\boldsymbol{E} S_\tau$*? What is* $\lim_{N\to\infty} \boldsymbol{E} S_{\tau\wedge N}$*?*

6.5. **American Options.** An American option is an option that can be exercised at any time chosen by the holder.

Definition 6.52. Let $G_0, G_1, \ldots, G_N$ be an adapted process. An *American option* with *intrinsic value* G is a security that pays G_σ at any finite stopping time σ chosen by the holder.

Example 6.53. An *American put* with strike K is an American option with intrinsic value $(K - S_n)^+$.

Question 6.54. *How do we price an American option? How do we decide when to exercise it? What does it mean to replicate it?*

Lecture 20 (10/18). Please enable your video if you can.

Wait

- Let G be an adapted process, and σ be a *finite* stopping time.
- Note $G_\sigma = \sum_{n=}^{N} G_n \mathbf{1}_{\sigma = n}$.
- Let $(X_0, (\Delta_n))$ be a self-financing portfolio, and X_n at time n be the wealth of this portfolio at time n.

Last time.

Definition 6.38. Consider a derivative security that pays G_σ at the random time σ. A self-financing portfolio with wealth process X is a replicating strategy if $X_\sigma = G_\sigma$.

Remark 6.39. If a replicating strategy exists, then at any time before σ, the wealth of the replicating strategy must equal the arbitrage free price V. That is, $\mathbf{1}_{\{n \leqslant \sigma\}} X_n = \mathbf{1}_{\{n \leqslant \sigma\}} V_n$.

Theorem 6.40. *The security with payoff G_σ (at the stopping time σ) can be replicated. The arbitrage free price is given by*

$$\boxed{V_n \mathbf{1}_{\{\sigma \geqslant n\}} = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n (D_\sigma G_\sigma \mathbf{1}_{\{\sigma \geqslant n\}})}$$

Remark 6.41. The only thing required for the proof of Theorem 6.40 is the fact that X_n is the wealth of a self-financing portfolio if and only if $D_n X_n$ is a $\tilde{\boldsymbol{P}}$ martingale.

$$\mathbb{1}_{\{n \leq \sigma\}} V_n = \mathbb{1}_{\{n \leq \sigma\}} \frac{1}{D_n} \tilde{E}_n (D_\sigma G_\sigma)$$

Proposition 6.42. *The wealth of the replicating portfolio (at times before σ) is uniquely determined by the recurrence relations:*

$$X_N \mathbf{1}_{\{\sigma=N\}} = G_N \mathbf{1}_{\{\sigma=N\}}$$

$$X_n \mathbf{1}_{\{\sigma\geqslant n\}} = G_n \mathbf{1}_{\{\sigma=n\}} + \frac{1}{1+r}\mathbf{1}_{\{\sigma>n\}} \tilde{\boldsymbol{E}}_n X_{n+1}\,.$$

If we write $\omega = (\omega', \omega_{n+1}, \omega'')$ with $\omega' = (\omega_1, \dots, \omega_n)$, then we know in the Binomial model we have

$$\tilde{\boldsymbol{E}}_n X_{n+1}(\omega) = \tilde{\boldsymbol{E}}_n X_{n+1}(\omega') = \tilde{p} X_{n+1}(\omega', 1) + \tilde{q} X_{n+1}(\omega', -1)\,.$$

(Recall X_n = wealth of rep port = AFP = V_n)

Pf: Rep port is self fin $\Rightarrow$ $D_n X_n$ is a $\tilde{P}$ mg

$\Rightarrow D_n X_n = \tilde{E}_n (D_{n+1} X_{n+1}) \Rightarrow X_n = \frac{1}{D_n} \tilde{E}_n (D_{n+1} X_{n+1})$

$= \frac{1}{1+r} \tilde{E}_n X_{n+1}$

$$\Rightarrow \mathbb{1}_{\{\tau \geq n\}} X_n = \underbrace{\mathbb{1}_{\{\tau \geq n\}}}_{\mathcal{F}_n\text{- meas}} \frac{1}{1+r} \tilde{E}_n X_{n+1}$$

$$= \frac{1}{1+r} \left(\tilde{E}_n \left(\mathbb{1}_{\{\tau = n\}} X_{n+1} + \mathbb{1}_{\{\tau > n\}} X_{n+1} \right) \right)$$

$$= \mathbb{1}_{\{\tau = n\}} \tilde{E}_n \frac{X_{n+1}}{1+r} + \frac{1}{1+r} \mathbb{1}_{\{\tau > n\}} \tilde{E}_n X_{n+1}$$

$$= \frac{1}{D_n} \mathbb{1}_{\{\tau = n\}} \tilde{E}_n \left(D_{n+1} X_{n+1} \right) + \frac{1}{1+r} \mathbb{1}_{\{\tau > n\}} \tilde{E}_n X_{n+1}$$

$$= \frac{1}{\cancel{D_n}} \mathbb{1}_{\{\sigma = n\}} \left(\cancel{D_n} X_n \right) + \qquad \|$$

$$n$$

$$= \mathbb{1}_{\{\sigma = n\}} X_\sigma +$$

$$= \mathbb{1}_{\{\sigma = n\}} G_\sigma + \frac{1}{1+r} \mathbb{1}_{\{\sigma > n\}} E_n X_{n+1}$$

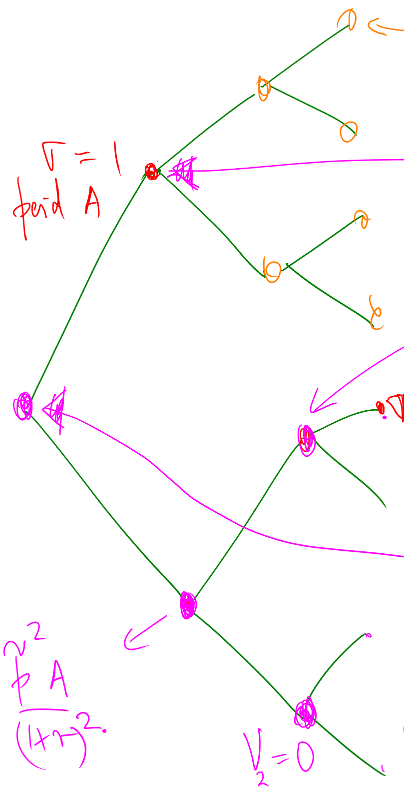
QED.

As before, we will use state processes to find practical algorithms to price securities.

Example 6.43. Let $A, U > 0$. The up-and-rebate option pays the face value A at the first time the stock price exceeds U (up to maturity time N), and nothing otherwise. Find an efficient way to compute the arbitrage free price of this option.

Say $d = \frac{1}{u}$, $S_0 = 1$, $U = uS_0 = u$, $N = 3$.

Option 1: Draw tree of all coin tosses up to $N = 3$ & price.



Don't need to find V_n

V_u

$V_1 = A$

$V_2 = \frac{\tilde{p} A}{1+r} + 0\tilde{q}$

$S_0 = 1$

$\tau = 3$, paid A

paid 0

paid 0

paid 0

$V_0 = \frac{1}{1+r}\left(\tilde{p} A + \tilde{q} \frac{\tilde{p}^2 A}{(1+r)^2}\right)$

$V_1 = \frac{\tilde{p}^2 A}{(1+r)^2}.$

$V_2 = 0$

2

Try same example: But look for $1_{\{n \leq \tau\}} V_n = 1_{\{n \leq \tau\}} f_n(S_n)$.

① $1_{\{\tau = N\}} V_N = \begin{cases} A & \text{if } S_N \geq u \\ 0 & \text{if not} \end{cases}$.

(Note $V_N \neq \begin{cases} A & \text{if } S_N \geq u \\ 0 & \text{o/w} \end{cases}$, but $1_{\{\tau = N\}} V_N = \begin{cases} A & \text{if } S_N \geq u \\ 0 & \text{o/w} \end{cases}$)

✓

② Say $\mathbb{1}_{\{\sigma \geq n+1\}} V_{n+1} = \mathbb{1}_{\{\sigma \geq n+1\}} f_{n+1}(S_{n+1})$ $(n \leq N-1)$

$\left(\text{Know } f_N(s) = A \mathbb{1}_{\{s \geq U\}} \right)$

See if we can find a fn f_n $\ni$ $\mathbb{1}_{\{\sigma \geq n\}} V_n = \mathbb{1}_{\{\sigma \geq n\}} f_n(S_n)$

Know $\underbrace{\mathbb{1}_{\{\sigma \geq n\}}}_{} V_n = \overbrace{\mathbb{1}_{\{\sigma = n\}} A} + \mathbb{1}_{\{\sigma > n\}} \frac{1}{1+r} \tilde{E}_n V_{n+1}$

$$= \mathbb{1}_{\{\tau \geq n\}} \left(A \, \mathbb{1}_{\{S_n \geq U\}} + \mathbb{1}_{\{S_n < U\}} \frac{1}{1+r} \tilde{E}_n \left[f(S_{n+1}) \right] \right)$$

$$= \mathbb{1}_{\{\tau \geq n\}} \left(A \, \mathbb{1}_{\{\underline{S_n} \geq U\}} + \mathbb{1}_{\{\underline{S_n} < U\}} \left(\frac{f_{n+1}(u \underline{S_n}) \, \tilde{p} + f_{n+1}(d \underline{S_n}) \, \tilde{q}}{1+r} \right) \right)$$

$$= \mathbb{1}_{\{\tau \geq n\}} \cdot \underline{f_n}(S_n), \quad \text{where}$$

recurrence relation!

$$\boxed{f_n(s) = A \, \mathbb{1}_{\{s \geq U\}} + \mathbb{1}_{\{s < U\}} \left(\frac{f_{n+1}(us) \, \tilde{p} + f_{n+1}(ds) \, \tilde{q}}{1+r} \right)}$$

Proposition 6.44. *Let $Y = (Y^1, \dots, Y^d)$ be a d-dimensional process such that for every n we have $Y_{n+1}(\omega) = h_{n+1}(Y_n(\omega), \omega_{n+1})$ for some deterministic function h_{n+1}. Let A_1, ..., $A_N \subseteq \mathbb{R}^d$, with A_N $\mathbb{R}^d$, and define the stopping time σ by*

$$\sigma = \min\{n \in \{0, \dots, N\} \mid Y_n \in A_n\}.$$

Let g_0, ... g_N be N deterministic functions on $\mathbb{R}^d$, and consider a security that pays $G_\sigma = g_\sigma(Y_\sigma)$. The arbitrage free price of this security is of the form $V_n \mathbf{1}_{\{\sigma \geqslant n\}} = f_n(Y_n)\mathbf{1}_{\{\sigma \geqslant n\}}$. The functions f_n satisfy the recurrence relation

$$f_N(y) = g_N(y)$$

$$f_n(y) = \mathbf{1}_{\{y \in A_n\}} g_n(y) + \frac{\mathbf{1}_{\{y \notin A_n\}}}{1+r}\Big(\tilde{p} f_{n+1}(h_{n+1}(y, 1)) + \tilde{q} f_{n+1}(h_{n+1}(y, -1))\Big)$$

Lecture 21 (10/20). Please enable video if you can

Last time: Up rebate option: pays face value A the **first** time stock price exceeds $\underline{\underline{U}}$

Found the AFP as $\mathbb{1}_{\{n \leq \tau\}} V_n = \mathbb{1}_{\{n \leq \tau\}} f_n(S_n)$

τ = time option pays / expires

$= \left(\min\{k \leq N \mid S_k \geq U\}\right) \wedge N$

[I found a recurrence relation for f_n in terms of f_{n+1}]

Proposition 6.44. *Let $Y = (Y^1, \dots, Y^d)$ be a d-dimensional process such that for every n we have $Y_{n+1}(\omega) = h_{n+1}(Y_n(\omega), \omega_{n+1})$ for some deterministic function h_{n+1}. Let $A_1, \dots, A_N \subseteq \mathbb{R}^d$, with $A_N \subseteq \mathbb{R}^d$, and define the stopping time σ by*

$$\sigma = \min\{n \in \{0, \dots, N\} \mid Y_n \in A_n\}.$$

Let $g_0, \dots g_N$ be N deterministic functions on $\mathbb{R}^d$, and consider a security that pays $G_\sigma = g_\sigma(Y_\sigma)$. The arbitrage free price of this security is of the form $V_n \mathbf{1}_{\{\sigma \geqslant n\}} = f_n(Y_n)\mathbf{1}_{\{\sigma \geqslant n\}}$. The functions f_n satisfy the recurrence relation

$$f_N(y) = g_N(y)$$

$$f_n(y) = \mathbf{1}_{\{y \in A_n\}} g_n(y) + \frac{\mathbf{1}_{\{y \notin A_n\}}}{1+r}\Big(\tilde{p} f_{n+1}(h_{n+1}(y,1)) + \tilde{q} f_{n+1}(h_{n+1}(y,-1))\Big)$$

Y^1 (superscript) $\longrightarrow$ coordinate

Y_1 (subscript) $\longrightarrow$ time.

For up rebate option.

$A_1 = [U, \infty)$

$d = 1$

$Y^1 = S^u$

6.4. **Optional Sampling.**

Theorem 6.45 (Doob's optional sampling theorem). *Let τ be a bounded stopping time and M be a martingale. Then* $\boldsymbol{E}_n M_\tau = M_{\tau\wedge n}$.

Remark 6.46. When dealing with finitely many coin tosses ($N < \infty$), *bounded stopping times* are the same as *finite stopping times*. When dealing with infinitely many coin tosses, the two notions are different.

Remark 6.47. When $N = \infty$ and τ is not bounded, the optional sampling theorem is still true if $X_{\tau\wedge k}$ is uniformly bounded in k.

Corollary 6.48. *If M is a martingale and τ is a bounded stopping time, then* $\boldsymbol{E}M_\tau = \boldsymbol{E}M_0$.

Note: Fix $\tau = n+1$ (is a stopping time)

$E_n M_\tau = E_n M_{n+1} = M_n$ (def of Mg)

$\| \text{OST}$

$M_n = M_{(n+1)\wedge n} = M_{\tau\wedge n}$ ✓

Say M is a mg $\longrightarrow$ fair game

$\Rightarrow EM_n = EM_0 = M_0$ ($\because M_0$ is not random).

$\hookrightarrow$ Q: Say we stop playing at some bdd stopping time $\underline{\underline{\tau}}$

then what is EM_τ?

<u>Claim</u>: $EM_\tau = EM_0$

Proof: By OST: $\overset{(n=0)}{E_0} M_{\tau\wedge 0} \overset{OST}{=} M_0$

$= EM_0$ ($\because M_0$ is not random)

Proof of Theorem 6.45 τ is a bdd stopping time

NTS $E_n M_\tau = M_{\tau \wedge n}$

Note $E_n M_\tau = E_n \left(\sum_{k=0}^{n} \underbrace{1_{\{\tau=k\}}}_{\mathcal{F}_n\text{-meas}} M_{\underbrace{\tau}_{k}} + \sum_{k=n+1}^{N} 1_{\{\tau=k\}} M_\tau \right)$

$$= \underbrace{E_n \left(\sum_{k=0}^{n} 1_{\{\tau=k\}} M_k \right)}_{\mathcal{F}_n\text{ - meas.}} + E_n \left(\sum_{k=n+1}^{N} 1_{\{\tau=k\}} M_\tau \right)$$

$$= \sum_{k=0}^{n} \mathbb{1}_{\{\tau=k\}} M_{n\wedge\tau} \qquad + \qquad \|$$

$$= \mathbb{1}_{\{\tau\le n\}} M_{\tau\wedge n} \qquad + \quad E_n\left(\sum_{k=n+1}^{N} \mathbb{1}_{\{\tau=k\}} M_\tau\right)$$

$$= \quad \| \qquad + \quad \sum_{k=n+1}^{N} E_n\Big(\underbrace{\mathbb{1}_{\{\tau=k\}} M_k}_{\mathcal{F}_k\text{- meas}}\Big) \qquad (k \geq n)$$

$$= \quad \| \quad + \sum_{k=n+1}^{N} E_n E_k \left(1_{\{\tau=k\}} M_k \right)$$

$$= \quad \| \quad + \sum_{k=n+1}^{N} E_n \left(1_{\{\tau=k\}} E_k \underline{\underline{M_k}} \right)$$

$$= \quad \| \quad + \sum_{k=n+1}^{N} E_n \left(1_{\{\tau=k\}} E_k M_N \right) \quad \left(\because M \text{ is a mg} \right)$$

$$= \text{〃} + \sum_{k=n+1}^{N} E_n \left(E_k \left(\mathbb{1}_{\{\tau=k\}} M_N \right) \right)$$

$$= \text{〃} + \sum_{k=n+1}^{N} E_n \left(\mathbb{1}_{\{\tau=k\}} M_N \right) \qquad \text{(tower prop.)}$$

$$= \text{〃} + E_n \left[\underbrace{\left(\sum_{k=n+1}^{N} \mathbb{1}_{\{\tau=k\}} \right)} M_N \right]$$

$$= \quad '' \quad + \underbrace{E_n\left(\mathbb{1}_{\{\tau > n\}} M_N \right)}_{\mathcal{F}_n\text{- meas.}}$$

$$= \quad '' \quad + \mathbb{1}_{\{\tau > n\}} E_n M_N$$

$$= \quad '' \quad + \mathbb{1}_{\{\tau > n\}} M_n =$$

$$= \mathbb{1}_{\{\tau \le n\}} M_{\tau \wedge n} + \mathbb{1}_{\{\tau > n\}} M_{\tau \wedge n} = M_{\tau \wedge n} \qquad \text{QED.}$$

Lecture 22 (10/22). Please enable video if you can

Last time: Doob OST: $\tau \to$ bdd stopping time
$M \to M_g$ $\Big\}$ $E_n M_{(\tau)} = M_{n \wedge \tau}$

(Note: For n'y many coin tosses,
OST is true if τ is bdd.
If $\tau < \infty$ a.s., then you typically need an extra cond.)

Consider a market with a few risky assets and a bank.

Proposition 6.49. *Suppose a market admits a risk neutral measure. If X is the wealth of a self-financing portfolio and τ is a bounded stopping time such that $X_0 = 0$, and $X_\tau \geqslant 0$, then $X_\tau = 0$. That is, there cant be an arbitrage opportunity at any bounded stopping time.*

(Under a RNM, X self fin $\iff D_n X_n$ is a $\tilde{P}$ mg)

NTS $X_\tau = 0$ a.s. Know $D_n X_n$ is a $\tilde{P}$ mg. (self fin).

$\Rightarrow$ OST $\tilde{E}(D_\tau X_\tau) = \tilde{E}_0(D_\tau X_\tau) = D_0 X_0 = 0$

Know $D_\tau X_\tau \geqslant 0$ & $\tilde{E}(D_\tau X_\tau) = 0$

$\Rightarrow D_\tau X_\tau = 0$ a.s. $\Rightarrow X_\tau = 0$ a.s.

QED

Question 6.50 (Gamblers ruin). *Suppose* $N = \infty$. *Let* ξ_n *be i.i.d. random variables with mean* 0, *and let* $X_n = \sum_1^n \xi_k$. *Let* $\tau = \min\{n \mid X_n = 1\}$. *(It is known that* $\tau < \infty$ *almost surely.) What is* $\boldsymbol{E} X_\tau$*? What is* $\lim_{N\to\infty} \boldsymbol{E} X_{\tau \wedge N}$*?*

$EX_\tau = 1 \quad (\neq 0)$

(Does not violate OST as τ need not be bdd)

Note X is a Mg

Game: iid coin tosses

Win \$1 if heads

Lose \$1 if tails.

Strategy: Play until you win \$$10^6$ & leave.

τ = first time to reach \$ 10^6 (a stopping time)

Know $\tau < \infty$ a.s.

Q: $EX_{\tau\wedge N} \overset{OST}{=} E_0 X_{\tau \wedge N} = X_0 = 0$

0 ($\tau \wedge N$ is a bdd stopping time)

$0 = \lim_{N\to\infty} EX_{\tau\wedge N} \neq EX_\tau = 1$

① the Game is fair!

② What is $E X_\tau$? (X_n = wealth at time n).

$$E X_\tau = 10^6$$

(Does not contradict OST: τ need not be bdd).

$\boxed{\text{③ Have we beaten the House?}}$

6.5. **American Options.** An American option is an option that can be exercised at any time chosen by the holder.

Definition 6.51. Let $G_0, G_1, \dots, G_N$ be an adapted process. An *American option* with *intrinsic value* G is a security that pays G_σ at any finite stopping time σ chosen by the holder.

Example 6.52. An *American put* with strike K is an American option with intrinsic value $(K - S_n)^+$.

Question 6.53. *How do we price an American option? How do we decide when to exercise it? What does it mean to replicate it?*

Strategy I: Let σ be a finite stopping time, and consider an option with (random) maturity time σ and payoff G_σ. Let V_0^σ denote the arbitrage free price of this option. The arbitrage free price of the American option *should be* $V_0 = \max_\sigma V_0^\sigma$, where the maximum is taken over all *finite* stopping times σ.

Definition 6.54. The *optimal exercise time* is a stopping time σ^* that maximizes $V_0^{\sigma^*}$ over all finite stopping times.

Definition 6.55. An optimal exercise time σ^* is called *minimal* if for every optimal exercise time τ^* we have $\sigma^* \leqslant \tau^*$.

Remark 6.56. The optimal exercise time need not be unique. (The *minimal* optimal exercise time is certainly unique.)

$V_0^\sigma \to$ AFP of a option that matures at σ & pays G_σ.

Know American option is worth more than any of these options.

Guess: AFP of American opt $= V_0 = \max_\sigma V_0^\sigma$

$\left(V_0 = V_0^{\sigma^*}\right)$.

Question 6.57. *Does this replicate an American option? Say σ^* is the optimal exercise time, and we create a replicating portfolio (with wealth process X) for the option with payoff G_{σ^*} at time σ^*. Suppose an investor cashes out the American option at time τ. Can we pay him?*

Strategy II: Replication. Suppose we have sold an American option with intrinsic value G to an investor. Using that, we hedge our position by investing in the market/bank, and let X_n be the our wealth at time n.

(1) Need $X_\sigma \geqslant G_\sigma$ for all finite stopping times σ. (Or equivalently $X_n \geqslant G_n$ for all n.)

(2) For (at-least) *one* stopping time σ^*, need $X_{\sigma^*} = G_{\sigma^*}$.

The arbitrage free price of this option is X_0.

Sell American opt for X_0 at time 0

Invest X_0 $\longrightarrow$ Wealth X_n at time n.

Lecture 23 (10/25). Please enable your video if you can.

Last time: American option $\longrightarrow$ Intrinsic value G.

Exercise at any stopping time τ (your choice)

Collect intr value, G_τ

Strategy I: Have an american option.

Recall it as an option with fixed (random) maturity time τ & payoff G_τ

AFP is V_0^τ

Price American option by selling to highest bidder.

i.e. $\underline{V_0} = \underbrace{\max_{\tau} V_0^{\tau}}$ over all finite stopping times τ.

Let τ^* be a stopping time for which $V_0 = V_0^{\tau^*} = \max_{\tau} V_0^{\tau}$

(optimal exercise time).

Strategy II: Sell an American op to an investor for X_0 \$.

investor can cash out at any finite stopping time τ.

(1) Need to ensure my wealth $X_\tau \geq G_\tau \quad \forall \tau$.

$$(\text{i.e. } \underline{\underline{X_n \geq G_n}} \quad \forall n \quad \text{a.s.})$$

(2) Also, for at least one stopping time $\underline{\underline{\tau^*}}$, need $\underline{X_{\tau^*} = G_{\tau^*}}$.

Question 6.57. *Does Strategy I replicate an American option? Say σ^* is the optimal exercise time, and we create a replicating portfolio (with wealth process X) for the option with payoff G_{σ^*} at time σ^*. Suppose an investor cashes out the American option at time τ. Can we pay him?*

(Strategy I doesn't immediately tell me how to invest X_0 so that we can replicate the American Option)

IOU: Use strat I to replicate.

Question 6.58. *Does Strategy II yield the same price as Strategy I? I.e. must* $X_0 = \max\{V_0^\sigma \mid \sigma$ *is a finite stopping time* $\}$*?*

Claim: Yes (Needs Proof IOU)

Question 6.59. *Is the wealth of the replicating portfolio (for an American option) uniquely determined?*

Replication:

Needed ① $X_n \geq G_n \quad \forall n.$

& ② For some τ^*, $X_{\tau^*} = G_{\tau^*}$

① $Y_n \geq G_n \quad \forall n.$

② $Y_{\tilde\tau^*} = G_{\tilde\tau^*}$ for some $\tilde\tau^*$

($\tilde\tau^*$ need not equal τ^*)

Must $X_n = Y_n$?

(Not immediately clear)

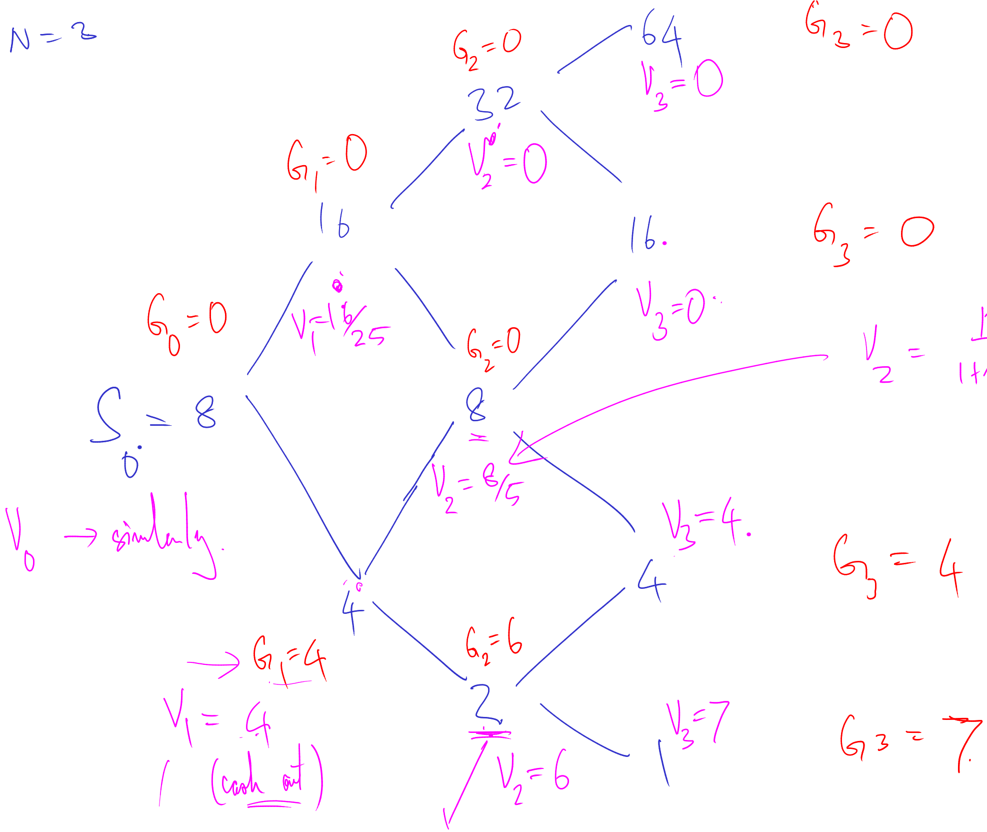
Question 6.60. *How do you find the minimal optimal exercise time, and the arbitrage free price? Let's take a simple example first.*

Eg: $u = 2$, $d = 1/2$, $r = 1/4$

$$\tilde{p} = \frac{1+r-d}{u-d} = \frac{5/4 - \frac{1}{2}}{3/2} = \frac{3/4}{3/2} = 1/2.$$

$$\tilde{q} = \qquad\qquad = 1/2.$$

~~America~~ American put strike $K = 8$. $(N = 3)$



Pink = What I think the AFP should be.

$$V_2 = \frac{1}{1+r}\left(\tilde{p}\,0 + \tilde{q}\cdot 4\right) = \frac{4}{5}\left(\frac{4}{2}\right) = \frac{8}{5}$$

$$V_1(1,*) = \frac{4}{5}\left(\frac{1}{2}\,\frac{8}{5}\right) = \frac{16}{25}$$

Cash out $\longrightarrow$ 6 (Better to cash out!)

Wait $\longrightarrow \frac{4}{5}\left(\frac{1}{2}\cdot 4 + \frac{1}{2}7\right) = \frac{2(11)}{5} = \boxed{\frac{22}{5}}$

$\approx$ 40 ...

Then Wait $\longrightarrow$ get $\frac{4}{5}\left(\frac{1}{2}\cdot\frac{8}{5} + \frac{6}{2}\right) = \frac{2}{5}\left(\frac{8}{5} + \frac{30}{5}\right) = \frac{76}{25}$

$\approx$ 3.0... \$.

Theorem 6.61. *Consider the binomial model with $0 < d < 1 + r < u$, and an American option with intrinsic value G. Define*

$$V_N = G_N\,, \qquad V_n = \max\Big\{\frac{1}{D_n}\tilde{\boldsymbol{E}}_n(D_{n+1}V_{n+1}), G_n\Big\}\,, \qquad \sigma^* = \min\{n \leqslant N \mid V_n = G_n\}\,.$$

Then V_n is the arbitrage free price, and σ^* is the minimal optimal exercise time. Moreover, this option can be replicated.

Remark 6.62. The above is true in any complete, arbitrage free market.

Remark 6.63. In the Binomial model the above simplifies to:

$$V_n(\omega) = \max\Big\{\frac{1}{1+r}\Big(\tilde{p}V_{n+1}(\omega', 1) + \tilde{q}V_{n+1}(\omega', -1)\Big), G_n(\omega)\Big\}\,, \qquad \text{where } \omega = (\omega', \omega_{n+1}, \omega'')\,, \quad \omega' = (\omega_1, \dots, \omega_n)\,.$$

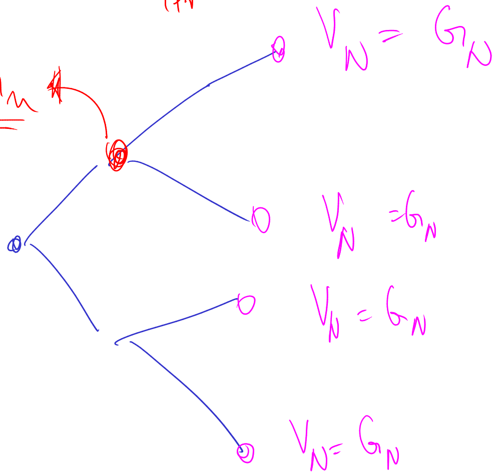
IOU a proof.

Lecture 24 (10/27): Please enable video if you can

Last time $\longrightarrow$ American options (proofs later)

Hold: Expected return: $\frac{1}{1+r}\tilde{E}_n V_{n+1} = \underbrace{\frac{1}{D_n}\tilde{E}_n(D_{n+1}V_{n+1})}$.

Exercise: G_n



Theorem 6.61. *Consider the binomial model with $0 < d < 1 + r < u$, and an American option with intrinsic value G. Define*

$$V_N = G_N\,, \qquad V_n = \max\Big\{\frac{1}{D_n}\tilde{\boldsymbol{E}}_n(D_{n+1}V_{n+1}), G_n\Big\}\,, \qquad \sigma^* = \min\{n \leqslant N \mid V_n = G_n\}\,.$$

(IOV → Pf)

Then V_n is the arbitrage free price, and σ^* is the minimal optimal exercise time. Moreover, this option can be replicated.

Remark 6.62. The above is true in any complete, arbitrage free market.

Remark 6.63. In the Binomial model the above simplifies to:

$$V_n(\omega) = \max\Big\{\frac{1}{1+r}\Big(\tilde{p}V_{n+1}(\omega', 1) + \tilde{q}V_{n+1}(\omega', -1)\Big), G_n(\omega)\Big\}\,, \qquad \text{where } \omega = (\omega', \omega_{n+1}, \omega'')\,, \quad \omega' = (\omega_1, \dots, \omega_n)\,.$$

Remark 6.64. We will prove Theorem 6.61 in the next section after proving the Doob decomposition.

Theorem 6.65. *Consider the Binomial model with $0 < d < 1 + r < u$, and a state process $Y = (Y^1, \dots, Y^d)$ such that $Y_{n+1}(\omega) = h_{n+1}(Y_n(\omega'), \omega_{n+1})$, where $\omega' = (\omega_1, \dots, \omega_n)$, $\omega = (\omega', \omega_{n+1}, \dots, \omega_N)$, and $h_0, h_1, \dots, h_N$ are N deterministic functions. Let $g_0, \dots, g_N$ be N deterministic functions, let $G_k = g_k(Y_k)$, and consider an American option with intrinsic value $G = (G_0, G_1, \dots, G_N)$. The* pre-exercise price *of the option at time n is $f_n(Y_n)$, where*

$$f_N(y) = g_N(y) \quad \textit{for } y \in \operatorname{Range}(Y_N), \qquad f_n(y) = \max\Big\{ g_n(y), \frac{1}{1+r}\Big(\tilde{p} f_{n+1}(h_{n+1}(y, u)) + \tilde{q} f_{n+1}(h_{n+1}(y, d))\Big)\Big\}, \quad \textit{for } y \in \operatorname{Range}(Y_n).$$

The minimal optimal exercise time is $\sigma^* = \min\{n \mid f_n(Y_n) = g_n(Y_n)\}$.

$+1$ $\quad -1$

Pf: Know $V_n = \max\left\{ G_n, \frac{1}{D_n} \tilde{E}_n(D_{n+1} V_{n+1}) \right\}$

$= \max\left\{ G_n, \frac{1}{1+r} \tilde{E}_n V_{n+1} \right\}.$

(2) Backward induction: Know $V_N = G_N = g_N(Y_N)$.

Set $f_N(y) = g_N(y) \quad \forall y \in \operatorname{Range}(Y_N)$

$$\left(\Rightarrow V_N = G_N = f_N(Y_N) \right).$$

③ Say $V_{n+1} = f_{n+1}(Y_{n+1})$. NTS $V_n = f_n(Y_n)$ (find f_n).

Know $V_n = \max\left\{ G_n, \frac{1}{1+r} \tilde{E}_n V_{n+1} \right\}$

$$= \max\left\{ g_n(Y_n), \frac{1}{1+r} \tilde{E}_n f_{n+1}(Y_{n+1}) \right\}.$$

$$= \max\left\{ g_n(Y_n), \frac{1}{1+r} \tilde{E}_n f_{n+1}\left(h_{n+1}(Y_n, \omega_{n+1}) \right) \right\}$$

$$= \max\left\{ g_n(\underline{Y_n}),\ \frac{1}{1+r}\left(\tilde{p}\, f_{n+1}\big(h_{n+1}(\underline{Y_n},+1)\big) + \tilde{q}\, f_{n+1}\big(h_{n+1}(\underline{Y_n},-1)\big)\right)\right\}.$$

Set $f_n(y) = \max\left\{ g_n(y),\ \frac{1}{1+r}\left(\tilde{p}\, f_{n+1}\big(h_{n+1}(y,+1)\big) + \tilde{q}\, f_{n+1}\big(h_{n+1}(y,-1)\big)\right)\right\}$

& get $V_n = f_n(Y_n)$

Also know $\tau^* = \min\{n \mid V_n = G_n\}$ QED.

Theorem 6.66. *Suppose the interest rate r is nonnegative. Let g be a convex function with $g(0) = 0$, and let $G_n = g(S_n)$. Consider an American option with intrinsic value $G_n = g(S_n)$. Then $\sigma^* = N$ is an optimal exercise time. That is, it is not advantageous to exercise this option early.*

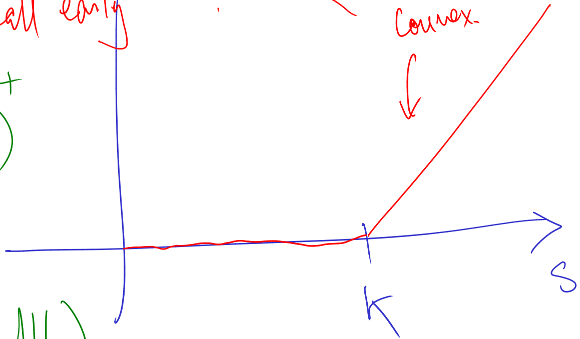
Corollary 6.67. *The arbitrage free price of an American call and European call are the same.*

Intrinsic value of American call : $g(S_n) = (S_n - K)^+$

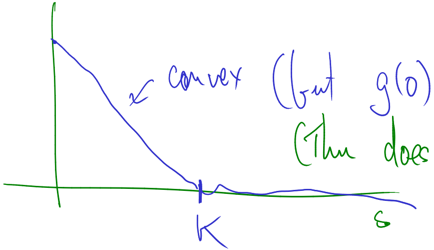
Thm $\Rightarrow$ No advantage to exercising American call early

& $g(0)=0$

Convex



Intrinsic val of American Put : $g(S_n) = (K - S_n)^+$

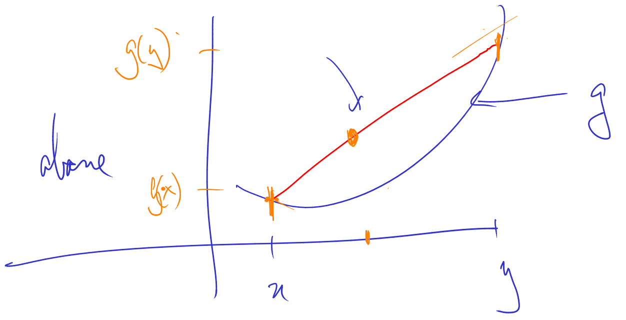


convex (but $g(0) \neq 0$)

(Thm does not apply).

Pf of 6.66 : (1) Convex :

By def $\rightarrow$ chord lies above the fn.



$\iff$ (2) $\forall\, x, y \in \mathrm{Domain}(g)$, $\theta \in [0,1]$,

From Calc :

f inc $\iff$

$$\theta g(x) + (1-\theta) g(y) \geq g(\overbrace{\theta x + (1-\theta) y}^{z})$$

2

③ Know $g(0)=0$ & g is convex.

$\Rightarrow \forall s \geq 0$, & $\theta \in [0,1]$, $g(\theta \cdot 0 + (1-\theta)s) \leq \theta g(0) + (1-\theta) g(s)$

$\Longleftrightarrow g((1-\theta)s) \leq (1-\theta) g(s)$. ... (*)

④ Know $V_n = \max\left\{ g(S_n), \underbrace{\frac{1}{1+r}\tilde{E}_n V_{n+1}}\right\}$.

Backward induction: Suppose $V_{n+1} \geq g(S_{n+1})$ (True for $n+1=N$)

Today : $\frac{1}{1+r} \tilde{E}_n V_{n+1} \geq \frac{1}{1+r} \tilde{E}_n g(S_{n+1}) = \tilde{E}_n \left(\frac{1}{1+r} g(S_{n+1}) \right)$

$\geq \tilde{E}_n g\left(\frac{S_{n+1}}{1+r} \right)$ (from ⊛ since $r \geq 0$

$\Rightarrow \frac{1}{1+r} \in [0, 1]$

& can choose $1-\theta = \frac{1}{1+r}$)

(Jensen's Inequality) $\geq g\left(\tilde{E}_n \frac{S_{n+1}}{1+r} \right)$

$= g(S_n)$ $\left[\because \tilde{E}_n S_{n+1} = (1+r) S_n \right]$

Know $V_n = \max\left\{ g(S_n), \frac{1}{1+r}\tilde{E}_n V_{n+1} \right\}$

$$\geq \max\left\{ g(S_n), \underbrace{\frac{1}{1+r}\tilde{E}_n\, g(S_{n+1})}_{\text{Just showed this is } \geq g(S_n)} \right\}$$

$\Rightarrow$ Holding the option longer <u>always</u> has a better expected return than exercising.

QED.

6.6. **Optimal Stopping.**

Definition 6.68. We say an adapted process M is a *super-martingale* if $\boldsymbol{E}_n M_{n+1} \leqslant M_n$.

Definition 6.69. We say an adapted process M is a *sub-martingale* if $\boldsymbol{E}_n M_{n+1} \geqslant M_n$.

Example 6.70. The discounted arbitrage free price of an American option is a *super-martingale* under the risk neutral measure.

Theorem 6.71 (Doob decomposition)**.** *Any adapted process can be uniquely expressed as the sum of a martingale and a predictable process that starts at* 0*. That is, if* X *is an adapted process there exists a unique pair of process* M, A *such that* M *is a martingale,* A *is predictable,* $A_0 = 0$ *and* $X = M + A$*.*

Proposition 6.72. *If X is a super-martingale, then there exists a unique martingale M and increasing predictable process A such that $X = M - A$.*

Proposition 6.73. *If X is a sub-martingale, then there exists a unique martingale M and increasing predictable process A such that $X = M + A$.*

Lecture 25 (10/29). Please enable video if you can

Rem: Have not yet proved

$$\underline{\underline{V_n}} = \max\left\{ G_n, \overbrace{\frac{1}{D_n}\widetilde{E}_n\left(D_{n+1}V_{n+1}\right)} \right\}$$

Gives the AFP of an american option (IOU)

6.6. Optimal Stopping.

Definition 6.68. We say an adapted process M is a *super-martingale* if $\boldsymbol{E}_n M_{n+1} \leqslant M_n$. (almost surely $\forall n$)

Definition 6.69. We say an adapted process M is a *sub-martingale* if $\boldsymbol{E}_n M_{n+1} \geqslant M_n$. (" " $\forall n$)

Example 6.70. The discounted arbitrage free price of an American option is a *super-martingale* under the risk neutral measure.

Mg : the fn $n \longmapsto E M_n$ is constant

Super mg : The fn $n \longmapsto E M_n$ is a decreasing fn of n

Sub mg : " " $n \longmapsto E M_n$ is an inc fn of n.

(Pf: If M is a super mg : $E_n M_{n+1} \leq M_n \Rightarrow E\underbrace{\left(E_n M_{n+1}\right)}_{E M_{n+1}} \leq E M_n$

QED.

Theorem 6.71 ($\boxed{\text{Doob decomposition}}$). *Any adapted process can be uniquely expressed as the sum of a martingale and a predictable process that starts at 0. That is, if X is an adapted process there exists a unique pair of process M, A such that M is a martingale, A is predictable, $A_0 = 0$ and $X = M + A$.*

Mg

Predictable, & $A_0 = 0$

Recall: A is a predictable process if A_n is $\mathcal{F}_{n-1}$ - measurable.

(In finance: Cash in bank $\to$ predictable process.)

Scratch work: Say $X_n = \underbrace{M_n}_{\text{Mg}} + \underbrace{A_n}_{\text{Pred},\ A_0=0}$

$$\Rightarrow \quad X_{n+1} = M_{n+1} + A_{n+1}$$

$$\Rightarrow \quad E_n X_{n+1} = E_n M_{n+1} + \underbrace{E_n A_{n+1}}$$

$$\underline{E_n X_{n+1} = M_n} + A_{n+1}$$

Want $A_0 = 0$.

$$\left. X_0 = M_0 + \underbrace{A_0}_{=0} \right\} \Rightarrow X_0 = \underline{M_0}$$

$$\boxed{X_1 = M_1 + A_1}$$

$$E X_1 = M_0 + \underline{\underline{A_1}} \Rightarrow A_1 = E X_1 - M_0$$

(know M_0, $A_0 = 0$ & A_1)

Since $X_1 = M_1 + A_1 \Rightarrow \boxed{M_1 = A_1 - X_1}$

Know $X_2 = M_2 + A_2 \Rightarrow \underline{E_1 X_2} = \underset{\smile}{M_1} + \underset{\smile}{A_2}$

$$\Rightarrow \boxed{A_2 = E_1 X_2 - M_1}$$

$$\Rightarrow M_2 = X_2 - (E_1 X_2) + M_1$$

Induction : ① Define $A_0 = 0$ & $M_0 = X_0$

② Given M_n & A_n, Define M_{n+1} & A_{n+1} by

ⓐ $A_{n+1} = E_n X_{n+1} - M_n$

ⓑ $M_{n+1} = M_n + X_{n+1} - E_n X_{n+1}$

③ Check : ⓐ Clearly A is predictable ($\because E_n X_{n+1} - M_n$ is $\mathcal{F}_n$ meas)

ⓑ Clearly M is a mg ($\because E_n M_{n+1} = M_n + E_n X_{n+1} - E_n X_n = M_n$)

& (c) $M_{n+1} + A_{n+1} = M_n + X_{n+1} - E_n X_{n+1} + E_n X_{n+1} - M_n$

$$= X_{n+1}$$ QED.

Uniqueness: (Proof above also shows uniqueness since A_0 & M_0 were unique & the choice of M_{n+1} & A_{n+1} that satisfies $X_{n+1} = M_{n+1} + A_{n+1}$ & $M \to$ mg, $A \to$ pred is also unique

QED

Proposition 6.72. *If X is a super-martingale, then there exists a unique martingale M and increasing predictable process A such that $X = M - A$.*

Proposition 6.73. *If X is a sub-martingale, then there exists a unique martingale M and increasing predictable process A such that $X = M + A$.*

Pf: Say X is a super Mg (i.e. $E_n X_{n+1} \leq X_n$)

Doob decomposition: Write $X = M + \tilde{A}$ (M is a mg, $\tilde{A}$ is predictable)

Set $A = -\tilde{A}$. $\Rightarrow X = \underbrace{M}_{\text{Mg}} - \underbrace{A}_{\text{Pred}}$

NTS: A is inc.: $X_{n+1} = M_{n+1} - A_{n+1}$

m

Condition on $\mathcal{F}_n$: $E_n X_{n+1} = E_n M_{n+1} - E_n A_{n+1}$

$(E_n X_{n+1} \leq X_n)$ $X_n \geq E_n X_{n+1} = M_n - A_{n+1}$

$X_n = M_n - A_n$

$\Rightarrow M_n - A_n \geq M_n - A_{n+1} \Rightarrow A_{n+1} \geq A_n$

$\Rightarrow A$ is increasing

QED.

Corollary 6.74. *If X is a super-martingale and τ is a bounded stopping time, then* $\boxed{\boldsymbol{E}_n X_\tau \leqslant X_{\tau \wedge n}.}$

Corollary 6.75. *If X is a sub-martingale and τ is a bounded stopping time, then* $\boldsymbol{E}_n X_\tau \geqslant X_{\tau \wedge n}$.

Recall: OST: If X is a mg & τ is a bdd stopping time

then $E_n X_\tau = X_{\tau \wedge n}$

$\Rightarrow$ Pf of 6.74: Say X is a super Mg

Doob decomp: Write $X = \underset{\text{Mg}}{\underset{\cup}{M}} - \underset{\text{Pred inc.}}{\underset{\cup}{A}}$

$\Rightarrow E_n X_\tau = E_n (M_\tau - A_\tau)$

$(a \wedge b = \min\{a, b\})$

$$= M_{\tau\wedge n} - E_n A_{\tau}$$

$$\le M_{\tau\wedge n} - E_n(A_{\tau\wedge n})$$

$(\because \tau\wedge n \le \tau$
$\Rightarrow A_{\tau\wedge n} \le A_\tau)$

$$= M_{\tau\wedge n} - A_{\tau\wedge n}$$

$(\because A_{\tau\wedge n}$ is $\mathcal{F}_n$-meas$)$

$$= X_{\tau\wedge n}.$$

QED

(Note

$$A_{\tau\wedge n} = \sum_{k=0}^{n} \underbrace{\underbrace{\mathbb{1}_{\{\tau=k\}}}_{\mathcal{F}_k \text{ meas}} \underbrace{A_k}_{\mathcal{F}_k\text{-meas}}}_{\mathcal{F}_n\text{-meas}} + \underbrace{\underbrace{\mathbb{1}_{\{\tau>n\}}}_{\mathcal{F}_n\text{-meas.}} \underbrace{A_n}_{\mathcal{F}_n\text{-meas}}}_{\mathcal{F}_n\text{-meas.}}$$

Theorem 6.76 (Snell). *Let G be an adapted process, and define V by*

$$V_N = G_N \qquad V_n = \max\{\boldsymbol{E}_n V_{n+1}, G_n\}.$$

Then V is the smallest super-martingale for which $V_n \geqslant G_n$.

Lecture 26 (11/1). Please enable video if you can.

Last time: Super Mg: $M_n \geq E_n M_{n+1}$

Sub Mg: $M_n \leq E_n M_{n+1}$

Doob Decomp: $X = \underset{\text{Mg}}{M} + \underset{\substack{\text{Predictable} \\ A_0 = 0}}{A}$ (A_{n+1} is $\mathcal{F}_n$-meas)

Cor: X is a super Mg $\Rightarrow$ $X = \underset{\text{Mg}}{M} - \underset{\substack{\text{Predictable} \\ A_0 = 0}}{A}$ & increasing

$(\Rightarrow E_n X_\tau \leq X_{\tau \wedge n})$

Theorem 6.76 (Snell). *Let G be an adapted process, and define V by*

$$V_N = G_N \qquad V_n = \max\{\boldsymbol{E}_n V_{n+1}, G_n\}.$$

Then V is the smallest super-martingale for which $V_n \geqslant G_n$.

Game: Can stop playing a game at any finite stopping time τ
collect reward G_τ

Pf: NTS ① V is a super Mg

② $V_n \geq G_n$

& ③ If W is any super Mg + $W \geq G$ then $W \geq V$.

(2): $V_n = \max\{G_n, E_n V_{n+1}\} \Rightarrow V_n \geq G_n.$

(1): $V_n = \max\{G_n, E_n V_{n+1}\} \Rightarrow V_n \geq E_n V_{n+1}$

$\Rightarrow V_n$ is a super mg.

Pf of (3): Let W be any super Mg. & $W \geq G$.

NTS $W \geq V$

Pf: Backward induction: (1) Certainly $W_N \geq G_N = V_N$ ✓

(2) Assume $W_{n+1} \geq V_{n+1}$

(a) $W_n \geq E_n W_{n+1}$ ($\because W$ is a super mg)

$\geq E_n V_{n+1}$ (induction Hyp)

(b) Already know $W_n \geq G_n$.

(a) & (b) $\Rightarrow$ $W_n \geq \max\{G_n, E_n V_{n+1}\} = V_n$ QED.

Proposition 6.77. *If W is any martingale for which $W_n \geqslant G_n$, and for one stopping time τ^* we have $\boldsymbol{E} W_{\tau^*} = \boldsymbol{E} G_{\tau^*}$, then we must have $W_{\tau^* \wedge n} = V_{\tau^* \wedge n}$, and $V_{\tau^* \wedge n}$ is a martingale.*

Pf: Note ① : $W_{\tau^*} = G_{\tau^*}$ $\quad$ ($\because$ $E W_{\tau^*} = E G_{\tau^*}$ & $W_{\tau^*} \geqslant G_{\tau^*}$)

Note ② : $W \geqslant \underline{V} \geqslant \underline{G}$ $\quad$ ($\because$ W a mg $\Rightarrow$ W is a super mg. & $W \geqslant G$ $\Rightarrow$ $W \geqslant V$)

smallest super mg $\geqslant G$.

Know: $W_{\tau^*} = G_{\tau^*} \Rightarrow W_{\tau^*} = V_{\tau^*} = G_{\tau^*}$.

$\Rightarrow$ $W_{\tau^*} = V_{\tau^*}$

(D.D. + OST $\to$ last time).

$\Rightarrow$ $W_{\tau^* \wedge u} \overset{\text{OST}}{=} E_u W_{\tau^*} = E_u V_{\tau^*} \leq V_{\tau^* \wedge u}$

Since we already know $W \geq V \Rightarrow W_{\tau^* \wedge u} = V_{\tau^* \wedge u}$.

Also $V_{\tau^* \wedge n}$ is a mg because $W_{\tau^* \wedge n}$ is a mg

$$\left(\text{OST} \Rightarrow E_n\left(W_{\tau^* \wedge (n+1)}\right) = W_{\tau^* \wedge (n+1) \wedge n} = W_{\tau^* \wedge n} \right)$$

QED.

Theorem 6.78. *Let $\sigma^* = \min\{n \mid V_n = G_n\}$. Then σ^* is the minimal solution to the optimal stopping problem for G. Namely, $\boldsymbol{E}G_{\sigma^*} = \max_\sigma \boldsymbol{E}G_\sigma$ where the maximum is taken over all finite stopping times σ. Moreover, if $\boldsymbol{E}G_{\tau^*} = \max_\sigma \boldsymbol{E}G_\sigma$ for any other finite stopping time τ^*, we must have $\tau^* \geqslant \sigma^*$.*

Remark 6.79. By construction $V_{\sigma^* \wedge n}$ is a martingale.

Pf of Thm: Know V is a super Mg.

Doob decomposition: $V = \underbrace{X}_{\text{Mg}} - \underbrace{A}_{\text{Pred, inc}}$

$A_0 = 0$.

Claim: $A_{\sigma^*} = 0$

(Note $\Rightarrow A_{\tau^* \wedge n} = 0 \Rightarrow V_{\sigma^* \wedge n} = \underbrace{X_{\sigma^* \wedge n}}_{\text{Mg}}$)

Pf of claim: $V_n = \max\{G_n, \underbrace{E_n V_{n+1}}\}$

$\tau^* = \min\{n \mid V_n = G_n\}$

$\Rightarrow$ for $n < \tau^*$, $V_n \neq G_n$ i.e. $V_n = E_n V_{n+1}$

More precisely $\mathbb{1}_{\{n<\tau^*\}} V_n = \mathbb{1}_{\{n<\tau^*\}} E_n V_{n+1}$

Παίρνω E_n

$$E_n\left(\mathbb{1}_{\{n<\tau^*\}} V_{n+1}\right) = E_n\left(\mathbb{1}_{\{n<\tau^*\}} X_{n+1} - \mathbb{1}_{\{n<\tau^*\}} A_{n+1}\right)$$

$$\Rightarrow \mathbb{1}_{\{n<\tau^*\}} E_n V_{n+1} = \mathbb{1}_{\{n<\tau^*\}} E_n X_{n+1} - \mathbb{1}_{\{n<\tau^*\}} E_n A_{n+1}$$

$$\Rightarrow \mathbb{1}_{\{n<\tau^*\}} V_n = \mathbb{1}_{\{n<\tau^*\}} X_n - \mathbb{1}_{\{n<\tau^*\}} A_{n+1}$$

$$(V = X - A) \Rightarrow \mathbb{1}_{\{n<\tau^*\}} A_{n+1} = \mathbb{1}_{\{n<\tau^*\}} A_n$$

$A_0 = 0 \quad \Rightarrow \quad A_{\sigma^*} = 0$

$$\left(\& \ \mathbb{1}_{\{n \le \sigma^*\}} A_n = 0 \right).$$

QED (Claim).

2020 Midterm 2 Q 3 b

$f(0) = 0 \qquad S_\tau \in \{0, M\}$

$f(M) = 1$

$$E f(S_\tau) = P(S_\tau = 0) \cdot \underbrace{f(0)}_{0} + P(S_\tau = M) \underbrace{f(M)}_{1}$$

$$= P(S_\tau = M)$$

Lecture 27 (11/8). Please enable video if you can

Last time: Optimal stopping problem

Play game. Can leave at any time $\longrightarrow$ reward G_n.

Stop at τ $\longrightarrow$ Get reward G_τ

Goal: $\max_\tau E\, G_\tau$

① Let $V_N = G_N$ & $V_n = \max\{G_n, E_n V_{n+1}\}$ $\longleftarrow$

Snell: V is the smallest super mg $\ni$ $V_n \geq G_n$ $\forall n$.

(2) Let W be a mg, $W_n \geq G_n$ $\forall n$.

and for some stopping time τ^*, $E W_{\tau^*} = E G_{\tau^*}$

Then $\underbrace{W_{\tau^* \wedge n}}$ $= V_{\tau^* \wedge n}$ $\left(\Longleftrightarrow \forall n,\ \mathbb{1}_{\{n \leq \tau^*\}} W_n = \mathbb{1}_{\{n \leq \tau^*\}} V_n \right)$

mg

(Last time).

(I.e. V is a mg before time τ^*).

Theorem 6.78. *Let $\sigma^* = \min\{n \mid V_n = G_n\}$. Then σ^* is the minimal solution to the* optimal stopping problem *for G. Namely, $\boldsymbol{E}G_{\sigma^*} = \max_\sigma \boldsymbol{E}G_\sigma$ where the maximum is taken over all finite stopping times σ. Moreover, if $\boldsymbol{E}G_{\tau^*} = \max_\sigma \boldsymbol{E}G_\sigma$ for any other finite stopping time τ^*, we must have $\tau^* \geqslant \sigma^*$.*

Remark 6.79. By construction $V_{\sigma^* \wedge n}$ is a martingale.

Pf: Last time: V is a super mg

Write $V = X - A$

X mg, A Predictable, increasing, $A_0 = 0$

Claim 1 (Last time): $A_{\tau^*} = 0$

Pf: $V_n = \max\{G_n, E_n V_{n+1}\}$ $\boxed{\tau^* = \min\{n \mid V_n = G_n\}}$

$\Rightarrow$ If $n < \tau^*$, then $V_n \neq G_n$

i.e. If $n < \tau^*$ $V_n = E_n V_{n+1}$

Now note $E_n V_{n+1} = E_n X_{n+1} - E_n A_{n+1}$ $\quad (n \leq \tau^*)$

$\Rightarrow$ $V_n = X_n - \underline{A_{n+1}}$ $\quad$ ($\because$ A is pred

X is a mg

& $n < \tau^* \Rightarrow E_n V_{n+1} = V_n$)

Already know $V_n = X_n - A_n$

$\Rightarrow \forall n < r^*$ must have $A_{n+1} = A_n$.

Since $A_0 = 0 \Rightarrow A_{n+1} = A_n = A_{n-1} \cdots = A_0 = 0$.

$\forall n < r^*$

i.e. $\forall m \leq r^*$, must have $A_m = 0 \Rightarrow$ Claim

QED

② $\Rightarrow$ $X_{\tau^*} = V_{\tau^*} = G_{\tau^*}$ $\Big(\because V_{\tau^*} = X_{\tau^*} - \underbrace{A_{\tau^*}}_{0}$

& def of $\tau^* \Big)$.

③ Claim: τ^* is a soln to the optimal stopping prob for G.

i.e. $\forall$ fin stopping times τ, $\quad EG_\tau \leq EG_{\tau^*}$

Pf: Note $G \leq V = X - A \quad \Rightarrow \quad X \geq V \geq G$.

$$\Rightarrow \quad E\,G_\tau \le E\,X_\tau \overset{OST}{=} X_0 \overset{OST}{=} E\,X_{\sigma^*} = E\,G_{\sigma^*}.$$

QED.

(4) **Claim:** If τ^* is any soln to the optimal stopping problem then $\tau^* \geq \sigma^*$.

Pf: Choose τ^* to be any soln to the optimal stopping problem.

i.e. $E\,G_{\tau^*} = \max_{\tau} E\,G_\tau$.

NTS $\boxed{\tau^* \geq \sigma^*}$.

$\underline{\underline{\text{Claim}}}$: $X_{\tau^*} = V_{\tau^*} = G_{\tau^*}$!

Pf: $E G_{\tau^*} \overset{\text{know}}{=} \max_{\sigma} E G_{\sigma} \overset{\text{know}}{=} E G_{\sigma^*}$

Know $E G_{\tau^*} \leq E X_{\tau^*}$ $\qquad (\because X \geq V \geq G)$

$\overset{OST}{=} X_0 \overset{OST}{=} E X_{\sigma^*} = E G_{\sigma^*} = E G_{\tau^*}$

$\Rightarrow E G_{\tau^*} = E X_{\tau^*}$

$$\Rightarrow \quad G_{\tau^*} = X_{\tau^*} \qquad \left(\because \; X_{\tau^*} \geq G_{\tau^*} \right)$$

Since $X \geq V \geq G \Rightarrow X_{\tau^*} = V_{\tau^*} = G_{\tau^*} \Rightarrow$ claim.

This implies $\tau^* \geq \sigma^*$ $\left(\because \; \sigma^* = \text{first time } V = G \right.$
$\left. \& \; V_{\tau^*} = G_{\tau^*} \right)$

QED

Theorem 6.80. *For any* $k \in \{0, \ldots, N\}$*, let* $\sigma_k^* = \min\{n \geqslant k \mid V_n = G_n\}$*. Then* $\boldsymbol{E}_k G_{\sigma_k^*} = \max_{\sigma_k} \boldsymbol{E}_k G_{\sigma_k}$*, where the maximum is taken over all finite stopping times* σ_k *for which* $\sigma_k \geqslant k$ *almost surely.*

(Pf: You check).

Lecture 28 (11/10) Please enable video if you can

Last time: Reward Process G

$$V_N = G_N \quad \& \quad V_n = \max\{G_n, E_n V_{n+1}\}$$

$$\tau^* = \min\{\forall n \mid V_n = G_n\}.$$

Showed V solves the optimal stopping problem

& τ^* = smallest optimal stopping time

Theorem 6.81. *Let $V = \cancel{M} - A$ be the Doob decomposition for V, and define $\tau^* = \max\{n \mid A_n = 0\}$. Then τ^* is a stopping time and is the* largest *solution to the optimal stopping problem for G.*

X

i.e. $E G_{\tau^*} \geq E G_\tau$ for any finite stopping time τ.

$X \to M_g$

$A \to$ 'Pred inc, $A_0 = 0$

Pf: (1) Check τ^* is a stopping time.

NOTE: In general $\max\{n \mid Y_n = b\}$ is NOT a stopping time if Y is adapted.

But for us: A is pred & inc & this makes τ^* a stopping time.

Note : $\{\tau^* = n\} = \underbrace{\underbrace{\{A_n = 0\}}_{\mathcal{F}_{n-1}\text{-meas}} \cap \underbrace{\{A_{n+1} > 0\}}_{\mathcal{F}_n\text{-meas}}}_{\mathcal{F}_n\text{-meas.}}$ ($\because$ A is inc)

(Recall: Predictable means
A_{n+1} is $\mathcal{F}_n$-meas)

2 Claim: $X_{\tau^*} = G_{\tau^*} = V_{\tau^*}$

Pf: ① $X_{\tau^*} = V_{\tau^*} - A_{\tau^*}$ & $A_{\tau^*} = 0 \Rightarrow X_{\tau^*} = V_{\tau^*}$.

② NTS $V_{\tau^*} = G_{\tau^*}$.

Say $\tau^* = n$. (Consider the event $\{\tau^* = n\}$)

$$V_{n+1} = X_{n+1} - A_{n+1}$$

$$\mathbb{1}_{\{\tau^* = n\}} E_n V_{n+1} = \mathbb{1}_{\{\tau^* = n\}} \left(X_n - \underbrace{A_{n+1}}_{>0} \right)$$

$$\Rightarrow \text{On } \{\tau^* = n\}, \quad E_n V_{n+1} < X_n = V_n + A_n = V_n + 0.$$

$$\Rightarrow \text{On } \{\tau^* = n\}, \quad E_n V_{n+1} < V_n$$

Know $V_n = \max\{G_n, E_n V_{n+1}\} \Rightarrow$ On $\{\tau^* = n\}$, $V_n = G_n$.

Since this holds $\forall n \Rightarrow V_{\tau^*} = G_{\tau^*}$.

QED (Claim 2)

Claim 3 τ^* is a soln to the optimal stopping problem.

Pf: Note for any stopping time τ,

$$E G_{\tau^*} = E X_{\tau^*} \overset{OST}{=} X_0 \overset{OST}{=} E X_\tau \geq E G_\tau$$

Q.E.D.

$$\boxed{\begin{array}{l} V = X - A,\ V \geq G \\ \Rightarrow X \geq V \geq G \end{array}}$$

<u>Claim 4</u>: τ^* is the largest solution to the optimal stopping problem.

Say τ^* is any solution to the optimal stopping problem

$$\Rightarrow \mathbb{E}\, G_{\tau^*} = \max_{\tau} \mathbb{E}\, G_{\tau} = \mathbb{E}\, G_{\tau^*} = \mathbb{E}\, X_{\tau^*}.$$

$$\Rightarrow \underline{\mathbb{E}\, G_{\tau^*}} = \mathbb{E}\, X_{\tau^*} \overset{OST}{=} X_0 \overset{OST}{=} \mathbb{E}\, X_{\tau^*}.$$

$$\Rightarrow G_{\tau^*} = X_{\tau^*} \quad \left(\because\ X_{\tau^*} \geq G_{\tau^*} \ \& \ \mathbb{E}\, X_{\tau^*} = \mathbb{E}\, G_{\tau^*} \right).$$

Know $X_\cdot \geq V_\cdot \geq G_\cdot \Rightarrow X_{\sigma^*} = V_{\sigma^*} = G_{\sigma^*}$

$\Rightarrow A_{\sigma^*} = 0 \Rightarrow \underline{\sigma^* \leq \tau^*}$

(def of τ^*)

QED.

6.7. **American options (with proofs).** Consider the N period binomial model with $0 < d < 1 + r < u$.

Proposition 6.82. *Any American option can be replicated. That is, consider an American option with intrinsic value G. There exists a self financing portfolio X such that:*

(1) *$X_n \geqslant G_n$ for all n*
(2) *For some stopping time σ^*, we have $X_{\sigma^*} = G_{\sigma^*}$.*

Pf: Let $\tilde{P}$ be the RNM.

Recall X is self financing $\iff D_n X_n$ is a $\tilde{P}$ mg

$$\left(D_n = (1+r)^{-n} \right).$$

Let $V_N = G_N$ & $V_n = \max\left\{ G_n, \frac{1}{D_n} \tilde{E}_n (D_{n+1} V_{n+1}) \right\}$

$$\tau^* = \min\{n \mid V_n = G_n\}.$$

Snell: $D_n V_n$ is the smallest super mg + $V_n \geq G_n \ \forall n$.

(& τ^* is the smallest ~~soln to the optimal stopping problem.~~ optimal exercise policy)

Doof decompose $D_n V_n$: Write $D_n V_n = \underbrace{D_n X_n}_{\tilde{P}\text{- Mg}} - \underbrace{A_n}_{\text{Pred, inc } A_0 = 0}$

X_n = wealth of a self fin Port ($\because D_n X_n$ is a $\tilde{P}$ mg).

Also, $D_n X_n = D_n V_n + A_n \geq D_n V_n \geq D_n G_n$

$\Rightarrow X_n \geq G_n \Rightarrow$ cond 1

Finally: $V_{\tau^*} = X_{\tau^*} = G_{\tau^*}$ (Snell)

$\Rightarrow$ Done

QED.

Proposition 6.83. *If X is the wealth of a replicating portfolio with* $\boxed{X_{\sigma^*} = G_{\sigma^*}}$. *Then σ^* is an optimal exercise policy. Moreover, if τ^* is any optimal exercise policy, then $X_{\tau^*} = G_{\tau^*}$*

Corollary 6.84 (Uniqueness)**.** *If X, and Y are wealth of two replicating portfolios for an American option with intrinsic value G, then for any optimal exercise time σ^* we must have $\mathbf{1}_{n \leqslant \sigma^*} X_n = \mathbf{1}_{n \leqslant \sigma^*} Y_n$.*

Pf: Know $D_n X_n$ is a $\tilde{P}$ - mg

$X_n \geqslant G_n$ & $X_{\tau^*} = G_{\tau^*}$.

NTS. $V^{\tau^*} = \max_{\tau^*} \underset{\sim}{V_0^{\tau}}$ $\left(V_0^{\tau} = \tilde{E}\left(D_\tau G_\tau\right)\right)$.

Lecture 29 (11/12). Please enabel video if you can

Last time: 1 American opt intrinsic value G

can be replicated! (i.e. $\exists$ a self fin port wealth X_n such that

1 $X_n \geq G_n$

& 2 $X_{\tau^*} = G_{\tau^*}$ for some stopping time τ^*

Proposition 6.83. *If X is the wealth of a replicating portfolio with $X_{\sigma^*} = G_{\sigma^*}$. Then σ^* is an optimal exercise policy. Moreover, if τ^* is any optimal exercise policy, then $X_{\tau^*} = G_{\tau^*}$*

Corollary 6.84 (Uniqueness). *If X, and Y are wealth of two replicating portfolios for an American option with intrinsic value G, then for any optimal exercise time σ^* we must have $\mathbf{1}_{n\leqslant\sigma^*} X_n = \mathbf{1}_{n\leqslant\sigma^*} Y_n$.*

Recall: ① An option with payoff G_σ at time σ has AFP $\tilde{E}(D_\sigma G_\sigma)$ at time 0

② Let $V_0^\sigma = \tilde{E}(D_\sigma G_\sigma) =$ AFP of the fixed unit option that pays G_σ at time σ.

③ Optimal exercise policy: $\sigma^* \ni V_0^{\sigma^*} = \max_\sigma V_0^\sigma$

Claim: $X_{\tau^*} = G_{\tau^*} \Rightarrow \tau^*$ is an optimal exercise policy

Pf: NTS $V_0^{\tau^*} \geq V_0^{\tau} \quad \forall \tau$ (finite stopping times)

i.e. NTS $\tilde{E} D_{\tau^*} G_{\tau^*} \geq \tilde{E}(D_\tau G_\tau)$

Pf: $\tilde{E}(D_{\tau^*} G_{\tau^*}) = \tilde{E}(D_{\tau^*} X_{\tau^*}) \overset{OST}{=\!=} \tilde{E}(D_0 X_0) \overset{OST}{=} \tilde{E}(D_\tau X_\tau)$

$\geq \tilde{E}(D_\tau G_\tau) = V_0^\tau$ QED

Conversely: Say τ^* is an optimal exercise policy

Then NTS $\underline{X_{\tau^*}} = G_{\tau^*}$.

Pf: $\tilde{E}\left(D_{\tau^*} G_{\tau^*}\right) = V_0^{\tau^*} = \max_{\sigma} V_0^{\sigma} = V_0^{\sigma^*} = \tilde{E}\left(D_{\sigma^*} G_{\sigma^*}\right)$

$= \tilde{E}\left(D_{\sigma^*} X_{\sigma^*}\right)$

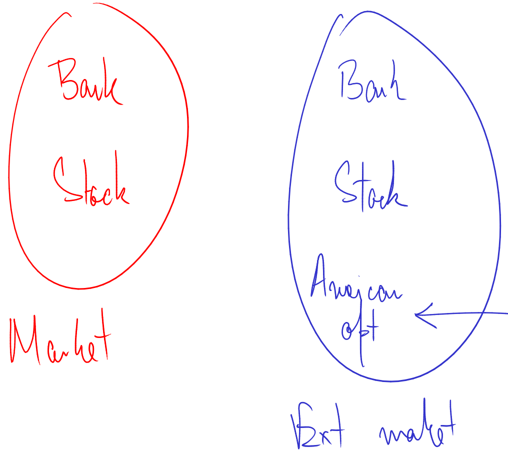
$\Rightarrow \tilde{E}\left(D_{\tau^*} G_{\tau^*}\right) = \tilde{E}\left(D_{\sigma^*} X_{\sigma^*}\right) \overset{OST}{=} \tilde{E}\left(D_0 X_0\right) \overset{OST}{=} \tilde{E}\left(D_{\tau^*} X_{\tau^*}\right)$

$\Rightarrow D_{\tau^*} G_{\tau} = D_{\tau^*} X_{\tau^*}$ $\left(\because D_{\tau^*} G_{\tau^*} \leq D_{\tau^*} X_{\tau^*} \ \& \ \tilde{E} \text{ are equal}\right)$.

QED!

Proposition 6.85. *Let $V_N = G_N$, and $V_n = \max\{G_n, D_n^{-1}\tilde{E}_n V_{n+1}\}$. Then V_n is the arbitrage free price of the American option. That is, the market remains arbitrage free if we are allowed to trade an American option at price V_n.*

$$\to V_n = \max\left\{ G_n \,,\, \frac{1}{D_n}\tilde{E}_n\left(D_{n+1} V_{n+1}\right)\right\}.$$



Price V_n at time n.

NTS: Ext is arb free.

(1) Say we buy one Am opt at time n (Price V_n)

Let $0 = \underbrace{V_n}_{\text{price of american opt}} - \underbrace{Y_n}_{\text{Borrowed \$}}$ (invest in cash/stock). $(V_n = Y_n)$

Sell option at time $\tau \geq n$ ($\tau \to$ stopping time).

Wealth at time $\tau = V_\tau - Y_\tau$

$$\Rightarrow \tilde{E}_n\left(D_\tau V_\tau - D_\tau Y_\tau\right) = \tilde{E}_n(D_\tau V_\tau) - D_n Y_n$$

($\because$ Y - self fin $\Rightarrow$ $D_n Y_n$ is a $\tilde{P}$ mg).

(Note $V_n = \max\left\{G_n, \frac{1}{D_n} E_n(D_{n+1} V_{n+1})\right\}$

$\Rightarrow D_n V_n \geq \tilde{E}_n(D_{n+1} V_{n+1})$

(i.e. $D_n V_n$ is a super mg under $\tilde{P}$)

$\Downarrow$

$$\hookrightarrow \Rightarrow \tilde{E}_n\left(D_\tau V_\tau - D_\tau Y_\tau\right) = \tilde{E}_n\left(D_\tau V_\tau\right) - D_n Y_n$$

$$\le D_n V_n - D_n Y_n \quad (\because D_n V_n \text{ is a } \tilde{P} \text{ supermartingale})$$

$$\le 0$$

$\Rightarrow$ no arb is possible!

(2) Say we sell an Am opt at time n.

$$\boxed{0 = \underbrace{Y_n}_{\text{invest in cash / stock.}} - \underbrace{V_n}_{\text{price of an opt}}}$$

Client exercises American option optimally (at time $\overbrace{\tau^* \vee n}^{\max\{\tau^*, n\}}$).

$$D_n V_n = D_n X_n - A_n \qquad (\text{Doob decomposition})$$

Know $A_{\rho^*} = 0$

Let $\tau = \rho^* \vee n$ & note $D_\tau V_\tau - D_n V_n = D_\tau X_\tau - D_n X_n - \underbrace{(A_\tau - A_n)}_{0}$

$$= D_\tau X_\tau - D_n X_n$$

$$\Rightarrow \tilde{E}_u\left(D_\tau V_\tau\right) = \tilde{E}_u\left(D_u V_u\right) = D_u V_u$$

$$\Rightarrow \tilde{E}_u\left(D_\tau Y_\tau - D_\tau V_\tau\right) = D_u Y_u - D_u V_u = 0$$

No arb is possible!

Q.E.D.

Lecture 30 (11/15) Please enable video if you can

7. Fundamental theorems of Asset Pricing

7.1. Markets with multiple risky assets.

(1) $\Omega = \{1, \dots, M\}^N$ is a probability space representing N rolls of M-sided dies, and p is a probability mass function on Ω.
(2) The die rolls need not be i.i.d.
(3) Consider a financial market with $d+1$ assets $S^0, S^1, \dots, S^d$. (S_n^k denotes the price of the k-th asset at time n.)
(4) For $i \in \{1, \dots, d\}$, S^i is an adapted process (i.e. S_n^i is $\mathcal{F}_n$-measurable).
(5) The 0-th asset S^0 is assumed to be a *risk free* bank/money market:
   (a) Let r_n be an adapted process specifying the interest rate at time n.
   (b) Let $S_0^0 = 1$, and $S_{n+1}^0 = (1 + r_n) S_n^0$. (Note S^0 is predictable.) in bank.
   (c) Let $D_n = (S_n^0)^{-1}$ be the discount factor (D_n dollars at time 0 becomes 1 dollar at time n).
(6) Let $\Delta_n = (\Delta_n^0, \dots, \Delta_n^d)$ be the position at time n of an investor in each of the assets $(S_n^0, \dots, S_n^d)$.
(7) The wealth of an investor holding these assets is given by $X_n = \Delta_n \cdot S_n \stackrel{\text{def}}{=} \sum_{i=0}^d \Delta_n^i S_n^i$.

S: $i \longrightarrow$ price of i^{th} asset.

$n \longrightarrow$ time n.

Dot product.

$\Delta_n^0 \longrightarrow$ Cash in bank at time n

$\Delta_n^i \longrightarrow$ # shares of i^{th} stock at time n.

Definition 7.1. Consider a portfolio whose positions in the assets at time n is Δ_n. We say this portfolio is *self-financing* if Δ_n is adapted, and $\boxed{\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}}$.

At time n $\longrightarrow$ position Δ_n

Wealth $\Delta_n \cdot S_n = \sum_{i=0}^{d} \Delta_n^i S_n^i$

Time $n+1$ $\longrightarrow$ Stock prices change from $S_n \longrightarrow S_{n+1}$

New wealth: $\Delta_n \cdot S_{n+1}$

↳ Change positions on the assets

$\boxed{\text{Rule}}$ → No external cash flows ($ in the market stays in the market).

New positions at time $n+1$ are $\Delta^0_{n+1}, \Delta^1_{n+1}, \dots \Delta^d_{n+1}$

No external cash flow ⟹ Wealth should be the same

$$\Rightarrow \boxed{\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}}$$

7.2. First fundamental theorem of asset pricing.

Definition 7.2. We say the market is arbitrage free if for any self financing portfolio with wealth process X, we have: $X_0 = 0$ and $X_N \geqslant 0$ implies $X_N = 0$ almost surely.

Definition 7.3. We say $\tilde{\boldsymbol{P}}$ is a *risk neutral measure* if $\boxed{\tilde{\boldsymbol{P}} \text{ is equivalent to } \boldsymbol{P}}$ and $\tilde{\boldsymbol{E}}_n(D_{n+1}S^i_{n+1}) = D_n S^i_n$ for every $i \in \{0, \dots, d\}$.

Theorem 7.4. *The market defined in Section 7.1 is arbitrage free if and only if there exists a risk neutral measure.*

$\tilde{P}$ is equiv to P if when $P(A) = 0 \iff \tilde{P}(A) = 0$

Note $\tilde{P}$ does not depend on i.

i.e. $\forall i \in \{1, \dots d\}$, $\tilde{E}_n(D_{n+1} S^i_{n+1}) = D_n S^i_n$

(for $i = 0$: By def $D_n = \dfrac{1}{S^0_n} \iff D_n S_n = 1$)

Diff from Binom : D_n is random

(D_n is a predictable process).

Lemma 7.5. *If $\tilde{\boldsymbol{P}}$ is a risk neutral measure, then the discounted wealth of any self financing portfolio is a $\tilde{\boldsymbol{P}}$-martingale.*

Proof that existence of a risk neutral measure implies no-arbitrage.

Pf of 7.5: Say $\tilde{P}$ is a RNM

$\Rightarrow \forall i \in \{0, \dots d\}, \quad \tilde{E}_n(D_{n+1} S^i_{n+1}) = D_n S^i_n$

Let X_n = wealth of any self fin port

$\Rightarrow X_n = \Delta_n \cdot S_n$ & Δ_n adapted

$$\boxed{\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}}$$

NTS $\tilde{E}_n(D_{n+1}X_{n+1}) = D_n X_n$

Note: $\tilde{E}_n\left(D_{n+1}X_{n+1}\right) = \tilde{E}_n\left(D_{n+1}\,\Delta_{n+1}\cdot S_{n+1}\right)$

$= \tilde{E}_n\left(D_{n+1}\,\Delta_n\cdot S_{n+1}\right)$ $\quad$ ($\because X_n$ is self fin)

$= \sum_{i=0}^{d} \Delta_n^i\,\tilde{E}_n\left(D_{n+1} S_{n+1}^i\right)$ $\quad$ ($\because \Delta_n$ is $\mathcal{F}_n$-meas)

$$= \sum_{i=0}^{d} \Delta_n^i \, D_n S_n^i \qquad (\because \text{Dfn of RNM})$$

$$= D_n \Delta_n \cdot S_n = D_n X_n$$

QED.

Pf that $\exists$ a RNM $\Rightarrow$ No arb!

Pf: $\tilde{P} \longrightarrow$ RNM.

Start with $X_0 = 0$ & X = wealth of a self fin port.

Suppose $X_N > 0$ NTS $X_N = 0$

Pf: Note $\tilde{E}(\underbrace{D_N X_N}) = D_0 X_0 = 0$

$(\because D_n X_n$ is a $\tilde{P}$ mg$)$.

Know $D_N X_N \geq 0$ $\underbrace{\tilde{P} \text{ a.s.}}$

$\Rightarrow D_N X_N = 0$ $(\tilde{P}$ a.s.$)$ $\Rightarrow D_N X_N = 0$ P (a.s.)

Lecture 31 (11/17). Please enable your video if you can.

Last time: Multiple assets S^0, S^1, ..., S^d

S^0 → Bank

$\underbrace{S^1, \ldots, S^d}_{\text{Stocks}}$

S^0_n ⟶ price of 1 share of M.M acct at time n.

int rate r_n: $S^0_{n+1} = (1+r_n) S^0_n$

Discount factor $D_n = \dfrac{1}{S^0_n}$

Notation : sub script $\longrightarrow$ time (n)

super script $i \longrightarrow i^{th}$ stock.

RNM : $\tilde{P}$ s.t. $\forall i \in \{1, \ldots d\}$, $\tilde{E}_n(D_{n+1} S^i_{n+1}) = D_n S^i_n$

(Note for $i=0$, $D_n S^0_n = D_{n+1} S^0_{n+1} = 1$

$\Rightarrow \tilde{E}_n(D_{n+1} S^0_{n+1}) = D_n S^0_n$)

Last time : FTAP 1 : ⓐ If a RNM exists then there is no arb.

(proved last time)

(b) No arb $\Longrightarrow$ $\exists$ a RNM

(need not be unique)

(IOU Proof $\to$ today).

Corollary 7.6. *Suppose the market has a risk neutral measure $\tilde{\boldsymbol{P}}$. Let V_N be a $\mathcal{F}_N$-measurable random variable and consider an security that pays V_N at time N. Then $V_n = D_n^{-1}\tilde{\boldsymbol{E}}_n(D_N V_N)$ is a arbitrage free price at time $n \leqslant N$. (i.e. allowing you to trade this security in the market with price V_n at time n keeps the market arbitrage free).*

Remark 7.7. We do not, however, know that the security can be replicated.

Pf: Last time: Self fin means $\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}$

Under $\tilde{P}$, $X \to$ self fin $\Rightarrow D_n X_n$ is a $\tilde{P}$ mg.

NTS: V_n = AFP at time n of the sec.

$\iff$ Extended market $(\underbrace{S^0 \,\&\, S^1, \dots S^d}\ \&\ \underbrace{V})$ is arb free.

new sec

By FTAP (part 1): Existence of a RNM $\Rightarrow$ No arb.

Will find a RNM for the extended market.

<u>Claim</u> $\tilde{P}$ is a RNM on the extended market!

Pf: ① Already know $D_n S_n^i$ is a $\tilde{P}$ mg $\forall i \in \{0, \dots d\}$.

② NTS $D_n V_n$ is a $\tilde{P}$ - mg

Note $V_n = \frac{1}{D_n} \tilde{E}_n (D_N V_N)$

$$\Rightarrow \underbrace{D_n V_n = \tilde{E}_n (D_N V_N)}$$

$$\Rightarrow \tilde{E}_n (D_{n+1} V_{n+1}) = \tilde{E}_n \left(\tilde{E}_{n+1} (D_N V_N) \right)$$

$$\overset{\text{tower}}{=} \tilde{E}_n (D_N V_N) = D_n V_n \quad \text{QED.}$$

Goal: Pf of converse $\to$ No arb $\Rightarrow \exists$ a RNM.

Lemma 7.8. *Suppose the market has no arbitrage, and X is the wealth process of a self-financing portfolio. If for any n, $X_n = 0$ and $X_{n+1} \geqslant 0$, then we must have $X_{n+1} = 0$ almost surely.*

(Def of No arb: $X_0 = 0$, $X_N \geqslant 0 \Rightarrow X_N = 0$

(self fin)

Lemma: $X_n = 0$, $X_{n+1} \geqslant 0 \Rightarrow X_{n+1} = 0$

Pf: If $\exists$ an arb between time n & $n+1$, then put \$ in bank at time $n+1$ & wait until N.

Lemma 7.9. *Suppose we find an equivalent measure* $\tilde{\boldsymbol{P}}$ *such that whenever* $\boxed{\Delta_n \cdot S_n = 0, \text{ we have } \tilde{\boldsymbol{E}}_n(\Delta_n \cdot S_{n+1}) = 0}$*, then* $\tilde{\boldsymbol{P}}$ *is a risk neutral measure.*

(Random $\Delta_n = (\Delta_n^0, \Delta_n^1, \dots \Delta_n^d)$

$$\Delta_n \cdot S_n = \sum_{i=0}^{d} \Delta_n^i S_n^i$$

① Lets check $D_n S_n^1$ is a $\tilde{P}$ mg.

NTS $\tilde{E}_n(D_{n+1} S^1_{n+1}) = D_n S_n$.

- At time n, buy 1 share of S^1 & borrow from bank i.e. $\Delta_n^1 = 1$ (1 share of S^1)

$\Delta_n^0 = -S_n^1 \cdot \left(\frac{1}{S_n^0}\right)$

$\Delta_n^i = 0 \quad \forall i \geq 1.$

$\Delta_n \cdot S_n = \Delta_n^0 S_n^0 + \Delta_n^1 S_n^1 + 0$

$= -\frac{S_n^1}{S_n^0} S_n^0 + 1 \cdot S_n^1 + 0 = 0$

By assumption: $\tilde{E}_n\left(\Delta_n \cdot S^1_{n+1}\right) = 0$

Compute $\Delta_n \cdot S_{n+1} = -\frac{S^1_n}{S^0_n} \cdot S^0_{n+1} + 1 \cdot S^1_{n+1} + 0$

$\Rightarrow \tilde{E}_n\left(\Delta_n \cdot S_{n+1}\right) = -\frac{S^1_n}{S^0_n} S^0_{n+1} + \tilde{E}_n S^1_{n+1} = 0$

$\left(\because D_n = \frac{1}{S^0_n} \ \& \ D_{n+1} = \frac{1}{S^0_{n+1}}\right)$

$\Rightarrow \tilde{E}_n S^1_{n+1} = S^1_n \cdot \frac{S^0_{n+1}}{S^0_n} \Rightarrow \tilde{E}_n\left(D_{n+1} S^1_{n+1}\right) = D_n S^1_n$

QED.

Lemma 7.10. *Suppose $\tilde{p}$ is a probability mass function such that $\tilde{p}(\omega) = \tilde{p}_1(\omega_1)\tilde{p}_2(\omega_1, \omega_2) \cdots \tilde{p}_N(\omega_1, \dots, \omega_N)$. If X_{n+1} is $\mathcal{F}_{n+1}$-measurable, then*

$$\tilde{\boldsymbol{E}}_n X_{n+1}(\omega) = \sum_{i=1}^{M} \tilde{p}_{n+1}(\omega', j) X_{n+1}(\omega', j)\,, \qquad \textit{where} \qquad \omega' = (\omega_1, \dots, \omega_n)\,, \omega = (\omega', \omega_{n+1}, \omega_{n+1}, \dots, \omega_N)$$

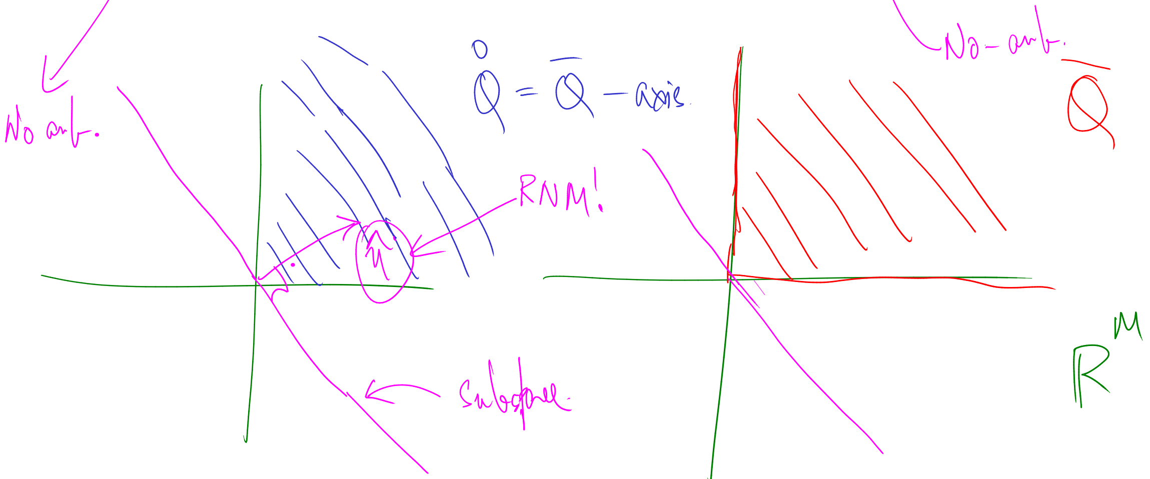
(Will remind you of this next time)

Lemma 7.11. *Define* $\bar{Q} \overset{\text{def}}{=} \{v \in \mathbb{R}^M \mid v_i \geqslant 0 \ \forall i \in \{1, \dots, M\}\}$, *and* $\mathring{Q} \overset{\text{def}}{=} \{v \in \mathbb{R}^M \mid v_i > 0 \ \forall i \in \{1, \dots, M\}\}$. *Let* $V \subseteq R^M$ *be a subspace.*

(1) $V \cap \bar{Q} = \{0\}$ *if and only if there exists* $\hat{n} \in \mathring{Q}$ *such that* $|\hat{n}| = 1$ *and* $\hat{n} \perp V$.

(2) *The unit normal vector* $\hat{n} \in \mathring{Q}$ *is unique if and only if* $V \cap \bar{Q} = \{0\}$ *and* $\dim(V) = M - 1$.

Remark 7.12. This can be proved using the *Hyperplane separation theorem* used in convex analysis.



Lecture 32 (11/19). Please enable your video if you can

FTAP 1: No arb $\Longleftrightarrow$ $\exists$ a RNM.

Simpler $\longrightarrow$ $\exists$ RNM $\Rightarrow$ No arb (done a few lectures ago)

Harder $\longrightarrow$ No arb $\Rightarrow$ $\exists$ RNM.

Last time: 1 Say $\tilde{P}$ is a measure & whenever $\Delta_n \cdot S_n = 0$

Wealth at time n

we have $\tilde{E}_n(\Delta_n \cdot S_{n+1}) = 0$

Then $\tilde{P}$ is a RNM.

Lemma 7.11. *Define* $\bar{Q} \stackrel{\text{def}}{=} \{v \in \mathbb{R}^M \mid v_i \geqslant 0 \ \forall i \in \{1, \ldots, M\}\}$, *and* $\mathring{Q} \stackrel{\text{def}}{=} \{v \in \mathbb{R}^M \mid v_i > 0 \ \forall i \in \{1, \ldots, M\}\}$. *Let* $V \subseteq R^M$ *be a subspace.*

(1) $V \cap \bar{Q} = \{0\}$ *if and only if there exists* $\hat{n} \in \mathring{Q}$ *such that* $|\hat{n}| = 1$ *and* $\hat{n} \perp V$.

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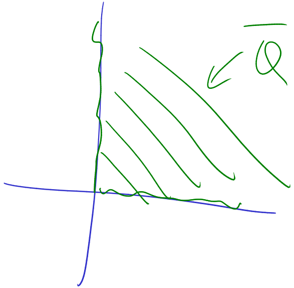
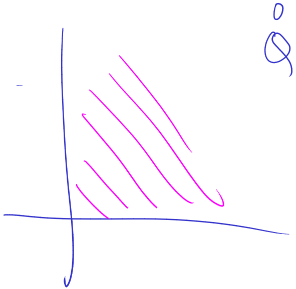
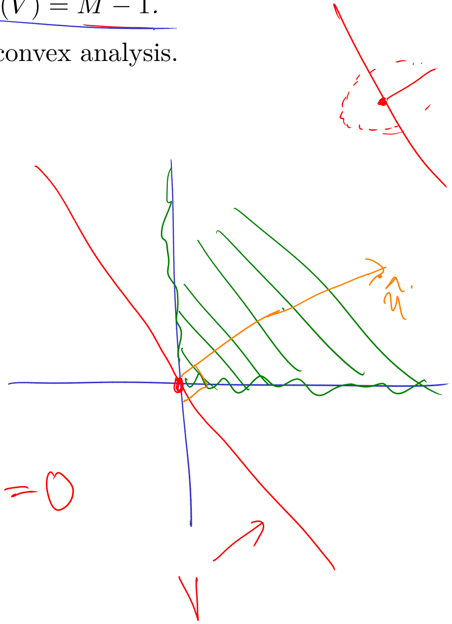






$\bar{Q}$

Recall: $\hat{n} \perp V$ means $\forall v \in V$, $\hat{n} \cdot v = 0$

$|\hat{n}| = \left(\sum \hat{n}_i^2\right)^{1/2}$.

Proof of Theorem 7.4 (No arbitrage implies existence of a risk neutral measure).

Assume : No arb. NTS : $\exists$ a RNM.

Case I:

N = 1 : Start with $X_0 = 0 = \Delta_0 \cdot S_0$ $\quad (\Delta_0 = (\Delta_0^0, \cdots \Delta_0^d) \in \mathbb{R}^{d+1})$

Let $V = \{\Delta_0 \cdot S_1 \mid \Delta_0 \cdot S_0 = 0\} \subseteq \mathbb{R}^M$

i.e. $V = \left\{ \begin{pmatrix} \Delta_0 \cdot S_1(1) \\ \Delta_0 \cdot S_1(2) \\ \vdots \\ \Delta_0 \cdot S_1(M) \end{pmatrix} \;\middle|\; \Delta_0 \cdot S_0 = 0 \right\}$

$\Delta_0 \cdot S_0 = \sum_{i=0}^{d} \Delta_0^i S_0^i$

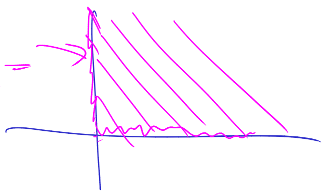
Recall : d stocks $S^1 \cdots S^d$ } d+1 assets

Bank S^0

Price changes by the roll of an M-sided die.

(i.e. $\Delta_0 \cdot S_1$ as a vector with i^{th} coordinate representing the wealth if the first die rolls i.)

Note: ① $V \subseteq \mathbb{R}^M$ is a subspace (You check it, quick).

② $V \cap \bar{Q} = \{0\}$ ($\because$ No out of $\bar{Q}$ = )

Hence: Lemma 7.11 $\Rightarrow \exists \hat{n} \in \mathring{Q}$ s.t. $\hat{n}(i) > 0$

($\hat{n}(i)$ = i^{th} coordinate of $\hat{n}$).

Let $\tilde{p}_1(i) = \dfrac{\hat{u}(i)}{\sum_{j=1}^{M} \hat{u}(j)}$ Claim $\tilde{p}_1(i) =$ RNP of 1^{st} die roll $= i$

(need denom to ensure $\sum_{i=1}^{M} \tilde{p}(i) = 1$)

Compute $\tilde{E}(\Delta_0 \cdot S_1)$ for any Δ_0 such that $\underbrace{\Delta_0 \cdot S_0 = 0}$

Note $\tilde{E}(\Delta_0 S_1) = \sum \tilde{p}(i) \cdot \Delta_0 \cdot S_1(i) = \sum \dfrac{\hat{u}(i)}{\left(\sum \hat{u}(j)\right)} \Delta_0 \cdot S_1(i)$

$$= \frac{1}{\left(\sum_j \hat{n}(j)\right)} \underbrace{\sum_i \hat{n}(i) \, \Delta_0 S_1(i)}_{\hat{n} \cdot \left(\Delta_0 S_1\right)}$$

$$= 0 \qquad (\because \hat{n} \text{ is a normal vector})$$

By lemma from last time $\Rightarrow \hat{P}$ is a RNM. QED $(N=1)$

Case 2: $N = 2$.

Suppose $\underline{\omega_1 = 1}$ (1^{st} die already rolled 1)

Start with $\Delta_1 \in \mathbb{R}^{d+1}$ $+$ $\Delta_1 \cdot S_1(1) = 0$

(i.e. wealth at time 1 if 1^{st} die is 1 $= 0$)

Let $V = \{ \Delta_1 \cdot S_2(1, \cdot) \mid \Delta_1 \cdot S_1(1) = 0 \}$

i.e. $V = \left\{ \begin{pmatrix} \Delta_1 \cdot S_2(1,1) \\ \Delta_1 \cdot S_2(1,2) \\ \vdots \\ \Delta_1 \cdot S_2(1,M) \end{pmatrix} \middle| \ \underline{\Delta_1 \cdot S_1(1) = 0} \right\}$

$\left.\begin{array}{l} ① \ V \subseteq \mathbb{R}^M \text{ is a subspace} \\ ② \text{ No arb} \Rightarrow V \cap \bar{Q} = \{0\}. \end{array}\right\}$ Lemma $\Rightarrow \exists \hat{u} \in \mathring{Q}$

Let $\tilde{p}_2(1, i) = \dfrac{\hat{u}(i)}{\sum_j \hat{u}(j)}$.

③ Say $\omega_1 = 1$ & $\Delta_1 \cdot S_1(1) = 0$

Compute $\tilde{E}_1(\Delta_1 \cdot S_2)(1) = \sum \tilde{p}_2(1, i)\ \Delta_1 S_2(1, i)$

$$= \frac{1}{\sum \hat{u}(j)} \left(\hat{u} \cdot \begin{pmatrix} \Delta_1 \cdot S_2(1,1) \\ \vdots \\ \Delta_1 \cdot S_2(1,M) \end{pmatrix} \right)$$

$$= 0 \qquad (\because \hat{u} \perp V).$$

Last time's lemma $\Rightarrow \tilde{P}$ is a RNM.

[Note: $\tilde{p}(\omega) = \tilde{p}_1(\omega_1)\, \tilde{p}_2(\omega_1, \omega_2) \cdots$].

Lecture 33 (11/22). Please enable video if you can.

So far: Binomial model $0 < d < 1+r < u \longrightarrow$ "complete & arb free"

Every security can be replicated.

FTAP 1: Arb free $\Longleftrightarrow$ $\exists$ a RNM.

7.3. Second fundamental theorem.

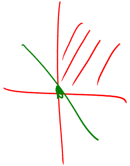
Definition 7.13. A market is said to be *complete* if every derivative security can be hedged.

Theorem 7.14. *The market defined in Section 7.1 is complete and arbitrage free if and only if there exists a* unique *risk neutral measure.*

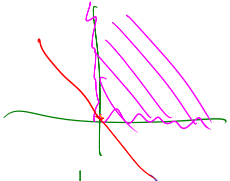
1 bank, d stocks $(S^0, S^1 \dots S^d)$

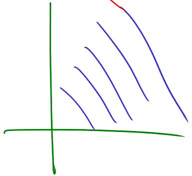
i.e. If V_N is the payoff of any security then $\exists$ a self fin port (X_n, Δ_n) such that $X_N = V_N$.

Wealth, trading strat



$(\Rightarrow \forall n \leq N$, X_n = unique AFP of the security$)$

Recall: $\mathbb{R}^M$: $\overline{Q} =$  $= \{ v \in \mathbb{R}^M \mid v_i \geq 0 \ \forall i \}$

$\mathring{Q} =$  $= \{ v \in \mathbb{R}^M \mid v_i > 0 \ \forall i \}$

Lemma: $V \subseteq \mathbb{R}^M$ a subspace.

(1) $V \cap \overline{Q} = \{0\} \iff \exists\, \hat{u} \in \mathring{Q}$ s.t. $\hat{u} \perp V$ & $|\hat{u}| = 1$

Use today $\longrightarrow$ (2) Say $V \cap \overline{Q} = \{0\}$. Then $\exists$ a unique $\hat{u} \in \mathring{Q}$ s.t. $\hat{u} \perp V$, $|\hat{u}| = 1 \iff \dim(V) = M-1$

Lemma 7.15. *The market is complete if and only if for every $\mathcal{F}_{n+1}$-measurable random variable X_{n+1}, there exists a (not necessarily unique) $\mathcal{F}_n$ measurable random vector $\Delta_n = (\Delta_n^0, \dots, \Delta_n^d)$ such that $X_{n+1} = \boxed{\Delta_n} \cdot S_{n+1}$.*

Pf: ① Say $n+1 = N$

Say market is complete.

X_N is some $\mathcal{F}_N$-meas RV.

Let $V_N = X_N$ = payoff of a security at time N

Complete $\Rightarrow$ can be replicated.

$\Rightarrow \exists$ a self fin port (X_n, Δ_n) + $X_N = V_N$.

a

Know ① Wealth at time n = $X_n = \Delta_n \cdot S_n = \sum_{i=0}^{d} \Delta_n^i S_n^i$

& ② $\Delta_n \cdot S_{n+1} = \underbrace{\Delta_{n+1} \cdot S_{n+1}}_{X_{n+1}}$

$\Rightarrow X_{n+1} = \Delta_n \cdot S_{n+1}$

$\Rightarrow V_N = X_N = \underline{\underline{\Delta_{N-1}}} \cdot S_N$.

Conversely: $\boxed{\text{Say } \forall\ \mathcal{F}_{n+1}\text{-meas RV } V_{n+1},\ \exists\ \Delta_n\ (\mathcal{F}_n\text{-meas}) \ni V_{n+1} = \Delta_n \cdot S_{n+1}}$

NTS: Market is complete.

Let V_N = payoff of some security

NTF (X_n, Δ_n) self fin $\ni X_N = V_N$.

Pf: By assumption $\exists\ \Delta_{N-1} \ni V_N = \Delta_{N-1} \cdot S_N$

Let $X_N = V_N$. Let $X_{N-1} = \Delta_{N-1} \cdot S_{N-1}$

Assumption $\Rightarrow \exists \Delta_{N-2} \left(\mathcal{F}_{N-2}\text{-meas} \right) \ni X_{N-1} = \Delta_{N-2} \cdot S_{N-1}$

Set $X_{N-2} = \Delta_{N-2} \cdot S_{N-2}$ & continue.

Self fin:

$$\Delta_{N-2} \cdot S_{N-1} \overset{\text{Want}}{=\!=} \Delta_{N-1} \cdot S_{N-1}$$

Note: $X_{N-1} = \Delta_{N-2} \cdot S_{N-1}$ & by choice $X_{N-1} = \Delta_{N-1} \cdot S_{N-1}$

$$\Rightarrow \quad \Delta_{N-2} \cdot S_{N-1} = \Delta_{N-1} \cdot S_{N-1}$$

(repeat $\Rightarrow$ self fin)

QED.

Proof of Theorem 7.14 NTS RNM unique $\iff$ complete & arb free.

Case 1: $N = 1$.

Let $\underline{V} = \{ \Delta_0 \cdot S_1 \mid \overbrace{\Delta_0 \cdot S_0 = 0} \}$

$= \left\{ \begin{pmatrix} \Delta_0 \cdot S_1(1) \\ \Delta_0 \cdot S_1(2) \\ \vdots \\ \Delta_0 \cdot S_1(M) \end{pmatrix} \in \mathbb{R}^M \;\middle|\; \Delta_0 \cdot S_0 = 0 \right\} \subseteq \mathbb{R}^M$ (subspace).

no restriction

Let $\underline{U} = \{ \underset{\sim}{\Delta_0} \cdot S_1 \mid \underline{\Delta_0 \cdot S_0 \in \mathbb{R}} \} \subseteq \mathbb{R}^M$ (subspace).

Recall: If $\exists\, \hat{u} \in \mathring{Q}$ (i.e. $\underline{\hat{u}_i > 0}$) $\ni$ $\hat{u} \perp V$

then can use $\hat{u}$ to make a RNM.

(Chose $\tilde{p}_1(i) \underset{\text{last time}}{=} \dfrac{\hat{u}_i}{\sum_{j=1}^{M} \hat{u}_j}$ got a RNM

& conversely $\tilde{p}_1(i)$ is the RN probability that 1^{st} die rolls i

then $\begin{pmatrix} \tilde{p}_1(1) \\ \vdots \\ \tilde{p}_1(M) \end{pmatrix} \in \mathring{Q}$ & is $\perp V$)

Say Market is complete & arb free.

$\Rightarrow V \cap \bar{Q} = \{0\}$ (follows since no arb)

& $U = \mathbb{R}^M$ (follows since the market is complete & Lemma 7.11).

In this case

$\dim(V) = \dim(U) - 1$ (on HW)

$\Rightarrow \dim(V) = M - 1$ (& $V \cap \bar{Q} = \{0\}$) $\Rightarrow \hat{n} \in \mathring{Q}$ is unique

rescale coordinates $\Rightarrow \tilde{p}_1$ is unique (i.e. RNM is unique).

Conversely: Suppose the RNM is unique.

Know if $\hat{u} \in \mathring{Q}$ & $\hat{u} \perp V$

then can rescale coordinates of $\hat{u}$ & get a RNM.

RNM unique $\Leftrightarrow$ $\hat{u}$ is unique $\Leftrightarrow$ $\dim(V) = M-1$

$\Rightarrow \dim(U) = M \overset{\text{Lemma}}{\Longrightarrow}$ Completeness!

Lecture 34 (11/29). Please enable video if you can.

Recall: FTAP 1: No arb $\iff$ Existence of a RNM.

FTAP 2: No arb & completeness $\iff$ Existence & uniq of a RNM

Every sec can be replicated.

S^0 Bank

S^1
$\vdots$
S^d Stocks.

$$D_n = \frac{1}{S^0_n}$$

7.4. **Examples and Consequences.**

Proposition 7.16. *Suppose the market model Section 7.1 is complete and arbitrage free, and let $\tilde{\boldsymbol{P}}$ be the unique risk neutral measure. If $D_n X_n$ is a $\tilde{\boldsymbol{P}}$ martingale, then X_n must be the wealth of a self financing portfolio.*

Remark 7.17. We've already seen in Lemma 7.5 that if a (not necessarily unique) risk neutral measure exists, then the discounted wealth of any self financing portfolio must be a martingale under it.

Remark 7.18. All pricing results/formulae we derived for the Binomial model that only relied on the analog of Proposition 7.16 will hold in complete arbitrage free markets.

Pf of Prop 7.16. Know $D_n X$ is a $\tilde{P}$ mg

NTS: X_n = wealth of a self fin port

i.e. NTF a trading strat Δ_n + $X_n = \Delta_n \cdot S_n$

self fin cond. & $\longrightarrow$ $\boxed{\Delta_n \cdot S_{n+1} = \Delta_{n+1} \cdot S_{n+1}}$

$$\Delta_n = (\Delta_n^0, \Delta_n^1 \dots \Delta_n^d), \quad \Delta_n \cdot S_n = \sum_{i=0}^{d} \Delta_n^i S_n^i = \text{total wealth}$$

Market is complete : $\Rightarrow \exists\, \Delta_{N-1}$ ($\mathcal{F}_{N-1}$ - meas) $\ni X_N = \Delta_{N-1} \cdot S_N$

<u>Claim</u> $X_{N-1} = \Delta_{N-1} \cdot S_{N-1}$

Pf: Know $X_N = \Delta_{N-1} \cdot S_N$

$$\Rightarrow \quad D_N X_N = \Delta_{N-1} \cdot (D_N S_N)$$

$$\Rightarrow \quad \tilde{E}_{N-1}(D_N X_N) = \Delta_{N-1} \cdot \left(\tilde{E}_{N-1}(D_N S_N) \right)$$

$$\Rightarrow \quad D_{N-1} X_{N-1} = \Delta_{N-1} \cdot (D_{N-1} S_{N-1})$$

(∵ $D_n X_n$ is a $\tilde{P}$ mg
∵ By def of $\tilde{P}$
$D_n S_n$ is a $\tilde{P}$ mg)

$$\Rightarrow \quad \underline{X_{N-1}} = \underline{\Delta_{N-1} \cdot S_{N-1}} \quad \cdots \quad (*)$$

QED

Recall: Def of completeness $\Rightarrow \exists \, \Delta_{N-2}$ s.t. $X_{N-1} = \Delta_{N-2} \cdot S_{N-1} \quad \cdots \quad (**)$

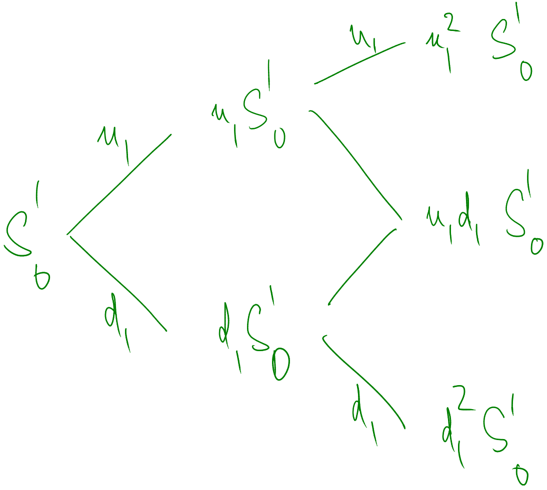
By above, get $X_{N-2} = \Delta_{N-2} \cdot S_{N-2}$

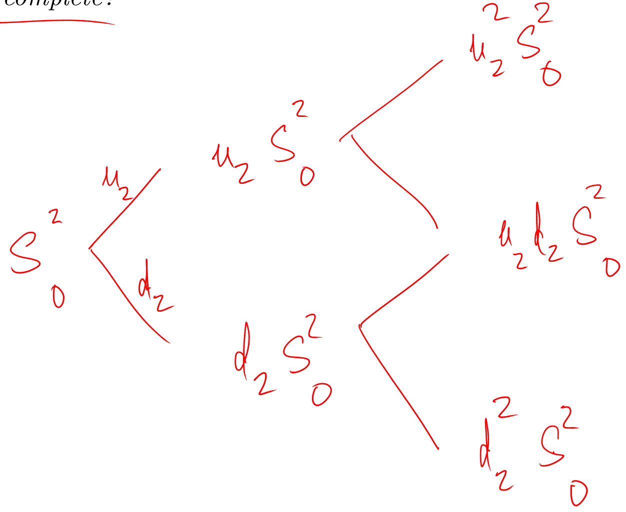
Keep going & get the trading strat (Δ_n)

Note: this trading strat is self fin

by equating $*$ & $**$ (& repeating)

Question 7.19. *Consider a market consisting of a bank with interest rate r, and two stocks with price processes S^1, S^2. At each time step we flip two independent coins. The price of the i-th stock ($i \in \{1, 2\}$) changes by factor u_i, or d_i depending on whether the i-th coin is heads or tails. When is this market arbitrage free? When is this market complete?*





Arb free $\iff$ $\exists$ a RNM.

$\tilde{p}, \tilde{q}$ RN Prob of heads & tails.

Need ① $\tilde{p} + \tilde{q} = 1$, $\tilde{p}, \tilde{q} > 0$

② $\tilde{E}_n S^1_{n+1} = (1+r) S^1_n \longrightarrow \tilde{E}_n S^1_{n+1} = \tilde{p} u_1 S^1_n + \tilde{q} d_1 S^1_n$

$= (\tilde{p} u_1 + \tilde{q} d_1) S^1_n$

$\overset{\text{Want}}{=} (1+r) S^1_n$

③ $\tilde{E}_n S^2_{n+1} = (1+r) S^2_n$

⇒ Shall have

$$① \quad \tilde{p} + \tilde{q} = 1$$

$$② \quad u_1 \tilde{p} + d_1 \tilde{q} = 1 + r$$

$$③ \quad u_2 \tilde{p} + d_2 \tilde{q} = 1 + r$$

$$\boxed{\text{In matrix form: } \begin{pmatrix} 1 & 1 \\ u_1 & d_1 \\ u_2 & d_2 \end{pmatrix} \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} 1 \\ 1+r \\ 1+r \end{pmatrix}} \quad \cdots \text{④}$$

Have a RNM $\Leftrightarrow$ we have > 0 ~~sd~~ solutions to $\circledast$.

Linear Alg: 3 eq & 2 unknowns $\rightarrow$ may not have a soln.

Will only have solutions if the system is consistent

e.g. if the 3rd eq is a comb of the first 2.

Finally: Need S^2 to be a combination of Bank & S^1

Rule of thumb: To get a complete & arb free market with d stocks, M.M., will need to roll a $d+1$ sided die

(Write down eq for RNM.
get (# stocks + 1) eqns in (# faces of the die)
unknowns)

Question 7.20. *Consider now repeated rolls of a 3-sided die and for $i \in \{1,2\}$, suppose $S^i_{n+1} = f_{i,j} S^i_n$, if $\omega_{n+1} = j$. How do you find the risk neutral measure? Find conditions when this market is complete and arbitrage free.*

$(j \in \{1, 2, 3\})$.

RNM: $\tilde{p}_i$ = Prob the die rolls i

System of eq:

(1) $\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 = 1$

(2) $\tilde{p}_1 f_{11} + \tilde{p}_2 f_{12} + \tilde{p}_3 f_{13} = 1+r$

(3) $\tilde{p}_1 f_{21} + \tilde{p}_2 f_{22} + \tilde{p}_3 f_{23} = 1+r$

$$\Leftrightarrow \quad \begin{pmatrix} 1 & 1 & 1 \\ b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} \underbrace{\begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \\ \tilde{\phi}_3 \end{pmatrix}} = \begin{pmatrix} 1 \\ 1+r \\ 1+r \end{pmatrix}$$

---

* Finding Δ:

Knows $\quad V_0 = \frac{1}{1+r} \tilde{E}_1 V_1 \quad \to$ find V_1

Q: $\Delta_0 = ?$

known

Choose Δ_0 so that $\Delta_0 \cdot S_1 = V_1$

$$\Delta_0^0 S_1^0(1) + \Delta_0^1 S_1^1(1) + \Delta_0^1 S_1^1(1) = V_1(1)$$

$$\sum_{i=0}^{d} \Delta_0^i S_1^i(j) = V_1(j) \leftarrow$$ $d+1$ equations (one for each j)

$$\begin{pmatrix} 1+r & f_{1,1} & f_{21} \\ 1+r & f_{1,2} & f_{22} \\ 1+r & f_{1,3} & f_{23} \end{pmatrix} \begin{pmatrix} \Delta_0^0 \\ \Delta_0^1 \\ \Delta_0^2 \end{pmatrix} = \begin{pmatrix} V_1(1) \\ V_1(2) \\ V_1(3) \end{pmatrix}$$

$d+1$ unknowns (each Δ_0^i)

8. Black-Scholes Formula

(1) Suppose now we can trade *continuously in time.*
(2) Consider a market with a bank and a stock, whose spot price at time t is denoted by S_t.
(3) The *continuously compounded interest rate* is r (i.e. money in the bank grows like $\partial_t C(t) = rC(t)$).
(4) Assume liquidity, neglect transaction costs (frictionless), and the borrowing/lending rates are the same.
(5) In the *Black-Scholes* setting, we model the stock prices by a *Geometric Brownian motion* with parameters α (the mean return rate) and σ (the volatility).
(6) The price at time t of a European call with maturity T and strike K is given by

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)),$$

$$\text{where} \quad d_\pm = \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big), \qquad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy.$$

(7) Can be obtained as the limit of the Binomial model as $N \to \infty$ by choosing:

$$r_{\text{binom}} = \frac{r}{N}, \qquad u = u_N = 1 + \frac{r}{N} + \frac{\sigma}{\sqrt{N}} \qquad d = d_N = 1 + \frac{r}{N} - \frac{\sigma}{\sqrt{N}}$$

9. **Recurrence of Random Walks**

- Let ξ_n be a sequence of i.i.d. coin flips with $\boldsymbol{P}(\xi_n = 1) = \boldsymbol{P}(\xi_n = -1) = 1/2$.
- Simple random walk: $S_n = \sum_1^n \xi_k$ (i.e. $S_0 = 0$, $S_{n+1} = S_n + \xi_{n+1}$).

Definition 9.1. The process S_n is recurrent at 0 if $\boldsymbol{P}(S_n = 0 \text{ infinitely often })$.

Question 9.2. *Is the random walk (in one dimension) recurrent at* 0*? How about at any other value?*

Question 9.3. *Say* ξ_n *are i.i.d. random vectors in* $\mathbb{R}^d$ *with* $\boldsymbol{P}(\xi_n = \pm e_i) = \frac{1}{2d}$. *Set* $S_n = \sum_1^n \xi_k$. *Is* S_n *recurrent at* 0*?*

Theorem 9.4. *The simple random walk in* $\mathbb{R}^d$ *is recurrent for* $d = 1, 2$ *and transient for* $d \geqslant 3$.

- Let $\tau_0 = \min\{n \mid S_n = 0\}$, be the first time S returns to 0.
- Let $\tau_1 = \min\{n \geqslant \tau_0 \mid S_n = 0\}$, be the first time after τ_0 that S returns to 0.
- Let $\tau_{k+1} = \min\{n \geqslant \tau_k \mid S_n = 0\}$, be the first time after τ_k that S returns to 0.

Lemma 9.5. *S is recurrent at 0 if and only if $\boldsymbol{P}(\tau_0 < \infty) = 1$.*

Lemma 9.6. *$\boldsymbol{P}(\tau_0 < \infty) = 1$ if and only if $\sum \boldsymbol{P}(S_n = 0) = \infty$.*

Proof.

Theorem 9.7. $\boldsymbol{P}(S_{2m} = 0) = O(1/m^{d/2})$. *Consequently, the random walk is* recurrent *for* $d \leqslant 2$, *and* transient *for* $d \geqslant 3$.

Lemma 9.8 (Sterling's formula)**.** *For large n, we have*

$$n! \approx \sqrt{2\pi} \exp\left(n \ln n - n + \frac{1}{2}\right).$$

Proof of Theorem 9.7 for $d = 1$:

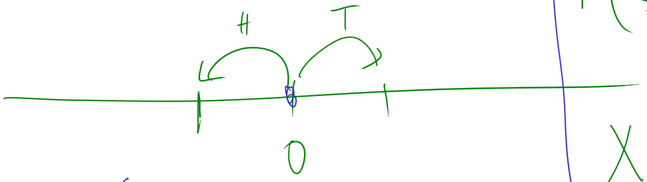
Lecture 35 (12/1). Please enable video if you can.

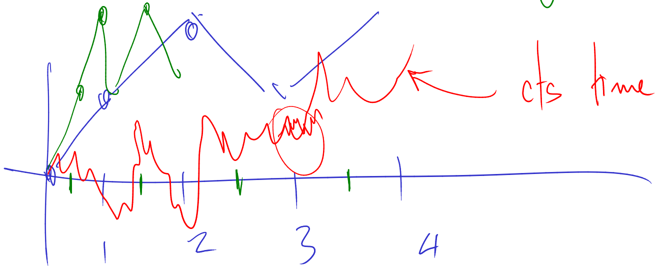
"Random walk in continuous time"

$\xi_n \longrightarrow$ iid

$P(\xi_n = 1) = P(\xi_n = -1) = \frac{1}{2}.$

$X_n = \sum_{k=1}^{n} \xi_k$





cts time limit

Cts time RW: flip a coin every $\frac{1}{N}$ seconds & take a step of size $\frac{1}{\sqrt{N}}$ left or right

Merton

8. Black-Scholes Formula

(1) Suppose now we can trade *continuously in time.*
(2) Consider a market with a bank and a stock, whose spot price at time t is denoted by S_t.
(3) The *continuously compounded interest rate* is r (i.e. money in the bank grows like $\partial_t C(t) = rC(t)$).

$C_t = C_0 \cdot e^{rt}$

(4) Assume liquidity, neglect transaction costs (frictionless), and the borrowing/lending rates are the same.
(5) In the *Black-Scholes* setting, we model the stock prices by a *Geometric Brownian motion* with parameters α (the mean return rate) and σ (the volatility).
(6) The price at time t of a European call with maturity T and strike K is given by

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)),$$

$\tau = T - t$

where $d_\pm = \dfrac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\dfrac{x}{K}\right) + \left(r \pm \dfrac{\sigma^2}{2}\right)\tau\right)$, $\quad N(x) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{x} e^{-y^2/2}\,dy$.

CDF of normal dist

(7) Can be obtained as the limit of the Binomial model as $N \to \infty$ by choosing:

$$r_{\text{binom}} = \frac{r}{N}, \qquad u = u_N = 1 + \frac{r}{N} + \frac{\sigma}{\sqrt{N}} \qquad d = d_N = 1 + \frac{r}{N} - \frac{\sigma}{\sqrt{N}}$$

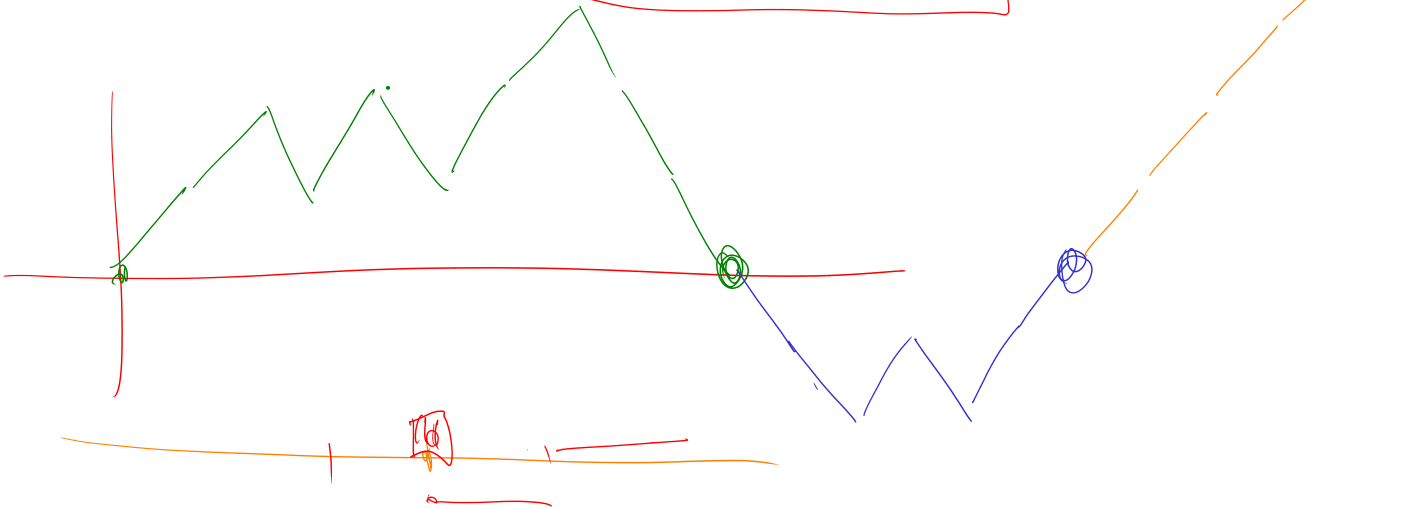
$N \to \infty!$

$u_N - d_N = \dfrac{2\sigma}{\sqrt{N}}$

9. Recurrence of Random Walks

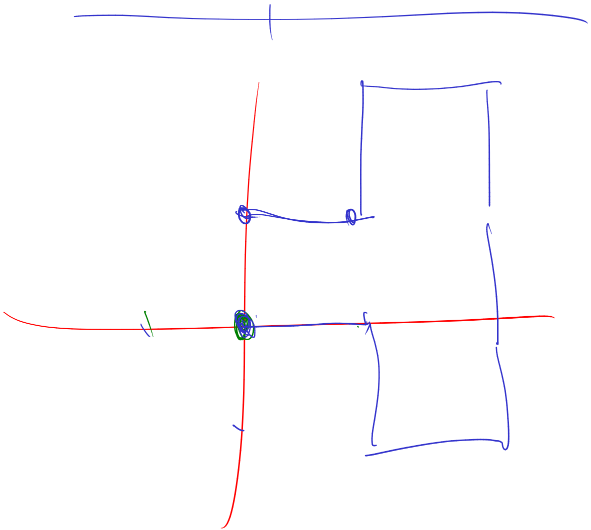
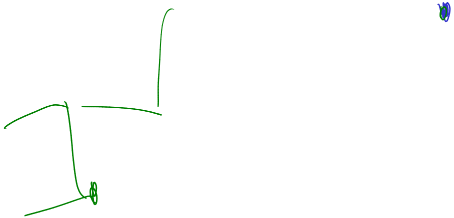
- Let ξ_n be a sequence of i.i.d. coin flips with $\boldsymbol{P}(\xi_n = 1) = \boldsymbol{P}(\xi_n = -1) = 1/2$.
- Simple random walk: $S_n = \sum_1^n \xi_k$ (i.e. $S_0 = 0$, $S_{n+1} = S_n + \xi_{n+1}$).

Definition 9.1. The process S_n is recurrent at 0 if $\boldsymbol{P}(S_n = 0 \text{ infinitely often }) = 1$



Question 9.2. *Is the random walk (in one dimension) recurrent at 0? How about at any other value?*

Question 9.3. *Say ξ_n are i.i.d. random vectors in $\mathbb{R}^d$ with $\boldsymbol{P}(\xi_n = \pm e_i) = \frac{1}{2d}$. Set $S_n = \sum_1^n \xi_k$. Is S_n recurrent at 0?*

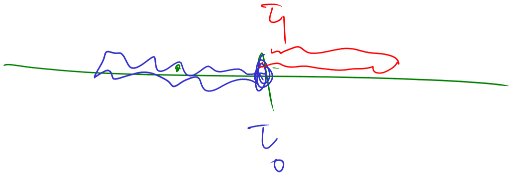
Theorem 9.4. *The simple random walk in* $\mathbb{R}^d$ *is recurrent for* $d = 1, 2$ *and transient for* $d \geqslant 3$.

Proof Below.

- Let $\tau_0 = \min\{n \mid S_n = 0\}$, be the first time S returns to 0.
- Let $\tau_1 = \min\{n \geqslant \tau_0 \mid S_n = 0\}$, be the first time after τ_0 that S returns to 0.
- Let $\tau_{k+1} = \min\{n \geqslant \tau_k \mid S_n = 0\}$, be the first time after τ_k that S returns to 0.

Lemma 9.5. *S is recurrent at 0 if and only if $\boxed{\boldsymbol{P}(\tau_0 < \infty) = 1}$.*

$P(S_n \text{ returns to } 0 \text{ infinitely often}) =$



Pf: (1) Say S is rec at 0

$\Rightarrow P(S_n \text{ returns to } 0 \text{ i.o.}) = 1 \Rightarrow P(S_n \text{ returns to } 0 \text{ once}) = 1$

$\Rightarrow P(\tau_0 < \infty) = 1$

(2) Conversely Say $P(\tau_0 < \infty) = 1$.

Note $P(\tau_1 < \infty) = P(\tau_0 < \infty)^2$

Odds of returning twice

= $\underbrace{\text{Odds of returning once}}_{P(\tau_0 < \infty)}$ & $\boxed{\underbrace{\text{then returning a second time}}_{P(\tau_0 < \infty)}}$

Ind

By ind: $P(\tau_n < \infty) = P(\tau_0 < \infty)^n = 1 \Rightarrow P(S_n \text{ returns to } 0 \text{ i.o.}) = 1.$

Lemma 9.6. $\boxed{\boldsymbol{P}(\tau_0 < \infty) = 1}$ *if and only if* $\boxed{\sum \boldsymbol{P}(S_n = 0) = \infty}$.

Proof.

i.e. $\boxed{\sum_{n=0}^{\infty} P(\cancel{P} S_n = 0) = \infty} \iff P(\tau_0 < \infty) = 1$

$\overset{\text{Lemma}}{\iff} P(S_n \text{ returns to } 0 \text{ i.o.}) = 1$

(i.e. S_n is recurrent at 0).

Lecture 36 (12/3). Please enable video if you can .

Thm (last time): $\xi_n \longrightarrow$ iid $\quad P(\xi_n = 1) = P(\xi_n = -1) = \frac{1}{2}$

$$S_n = \sum_1^n \xi_k \qquad S_0 = 0$$

S is rec at at 0 if

$$P(\{S_n = 0 \text{ for infinitely many } n\}) = 1$$

Thm (last time): In dim $d = 1$ or 2, the RW is rec.

In dim $d \geq 3$ the RW is NOT rec.

- Let $\tau_1 = \min\{n > 0 \mid S_n = 0\}$, be the first time S returns to 0.
- Let $\tau_2 = \min\{n > \tau_1 \mid S_n = 0\}$, be the first time after τ_1 that S returns to 0.
- Let $\tau_{k+1} = \min\{n > \tau_k \mid S_n = 0\}$, be the first time after τ_k that S returns to 0.

Lemma 9.5. *S is recurrent at 0 if and only if $\boldsymbol{P}(\tau_1 < \infty) = 1$.*

Pf Last time.

Lemma 9.6. $\boldsymbol{P}(\tau_0 < \infty) = 1$ *if and only if* $\sum_{n=0}^{\infty} \boldsymbol{P}(S_n = 0) = \infty$.

Proof.

(By 9.5, $\sum P(S_n = 0) = \infty \iff S$ is recurrent at 0).

Pf: (1) $E(\#\text{times } m \ni S_m = 0) = \sum_{n=0}^{\infty} P(S_n = 0)$

($\because$ $E(\#\text{times } m \ni S_m = 0) = E\left(\sum_m \mathbb{1}_{\{S_m = 0\}}\right)$

$= \sum_m E \mathbb{1}_{\{S_m = 0\}} =$

② $E(\#\text{times } m \ni S_m = 0) = \dfrac{1}{1 - P(\tau_1 < \infty)}$

Pf: $E(\#\text{times } m \ni S_m = 0) = E\left(\overbrace{\sum_{n=1}^{\infty} \mathbb{1}_{\{\tau_n < \infty\}}}\right)$

$= \sum_{n=1}^{\infty} P(\tau_n < \infty) = \sum_{n=1}^{\infty} P(\tau_1 < \infty)^n$

$= \dfrac{P(\tau_1 < \infty)}{1 - P(\tau_1 < \infty)}$ $\quad \left(\because P(\tau_2 < \infty) = P(\tau_1 < \infty)^2\right)$

Equate 1 & 2 : $E(\#\text{ times } n \ni S_n = 0) = \sum P(S_n = 0) = \dfrac{P(\tau_1 < \infty)}{1 - P(\tau_1 < \infty)}$

$\Rightarrow \quad \sum P(S_n = 0) = \infty \iff P(\tau_1 < \infty) = 1$

QED.

Theorem 9.7. $\boldsymbol{P}(S_{2m} = 0) = \boxed{O(1/m^{d/2})}$. *Consequently, the random walk is* recurrent *for* $d \leqslant 2$, *and* transient *for* $d \geqslant 3$.

S_n = RW in d dim.

Assume $P(S_{2n} = 0) = O\left(\frac{1}{n^{d/2}}\right)$.

Then S is rec at 0 $\iff \sum P(S_n = 0) = \infty$

$\iff \sum P(S_{2n} = 0) = \infty$.

$\iff \sum_{n=1}^{\infty} \frac{1}{n^{d/2}} = \infty$

Note Comparison test: $\sum \frac{1}{n^s} < \infty \iff s > 1$

$\therefore$ S is rec at $0 \iff d/2 \leq 1 \iff d \leq 2$ QED.

Compute $P(S_{2n} = 0)$.

$d = 1$: $P(S_{2n} = 0) = \binom{2n}{n} \cdot \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^n$

$= \frac{1}{2^{2n}} \binom{2n}{n} = \frac{1}{2^{2n}} \frac{(2n)!}{(n!)^2} \quad \ldots (*)$

Lemma 9.8 (Sterling's formula). *For large n, we have*

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n = \sqrt{2\pi}\exp\left(n\ln n - n + \frac{\ln n}{2}\right).$$

(Intuition: $n! = \prod_{k=1}^{n} k = e^{\sum_{k=1}^{n} (\ln k)}$

$$\sum_{k=1}^{n} \ln k \approx \int_1^n \ln x \, dx = \underbrace{n \ln n - n}\,)$$

Using Sterling on $\circledast$ $\Rightarrow$ $P(S_{2n}=0) \approx \frac{1}{2^{2n}} \cdot \left(\frac{\sqrt{2\pi\, 2n}\left(\frac{2n}{e}\right)^{2n}}{\left[\sqrt{2\pi\, n}\left(\frac{n}{e}\right)^n\right]^2}\right).$

~~*Proof of Theorem 9.7 for d = 1:*~~

$$\approx \frac{1 \quad \sqrt{2}}{2^{2n} \sqrt{2\pi n}} \frac{\sqrt{n}}{n} \frac{(2n)^{2n}}{n^{2n}}$$

$$\approx \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{n}} \qquad \text{as claimed}$$

$\Rightarrow$ Thm for $d = 1$

QED

Remark 9.9. Recall the Gambler's ruin example (Question 6.50): Let ξ_n be i.i.d. random variables with mean 0, and let $X_n = \sum_1^n \xi_k$. Let $\tau = \min\{n \mid X_n = 1\}$. Theorem 9.7 proves $\tau < \infty$ almost surely. We proved earlier $\boldsymbol{E} X_\tau = 1$ and $\lim_{N \to \infty} \boldsymbol{E} X_{\tau \wedge N} = 0$.

Theorem 9.10. *Consider the Gamblers ruin example, with $\tau = \min\{n \mid X_n = 1\}$. Then*

$$\boxed{\boldsymbol{E}\tau = \infty} \quad \text{and} \quad \boldsymbol{P}(\tau = 2n-1) = (-1)^{n-1}\binom{1/2}{n} \approx \frac{C}{n^{3/2}}$$

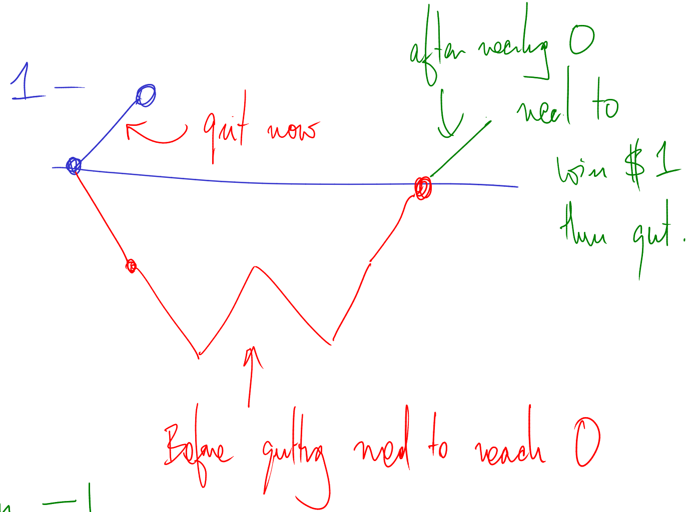
Remark 9.11. Let $M_n = \min\{X_{\tau\wedge k} \mid k \leqslant n\}$. Then $\boldsymbol{E}M_\tau = -\infty$. Thus, this strategy will take (on average) an infinite time before you win \$1. During that time your expected maximum loss is $-\infty$.

Lemma 9.12. *Let* $F(x) = \boldsymbol{E}x^\tau$. *Then* $\boxed{F(x) = \frac{1}{x}(1 - \sqrt{1-x^2})}$.

Pf:

$$F(x) = E\, x^{\tau}$$

$$= \frac{1}{2} x^{1} + \frac{1}{2} E\, x^{\tau' + \tau'' + 1}$$



τ' = first time X_n reaches 0 starting from -1

$\tau'' =$ first time after τ', X_n reaches $+1$

dist of τ' = dist of τ'' = dist of τ & τ'' & τ' are ind

$$\Rightarrow E x^{\tau} = \frac{1}{2} x + \frac{1}{2} \quad x E x^{\tau'} E x^{\tau''}$$

$$= \frac{1}{2} x + \frac{x}{2} \left(E x^{\tau} \right)^2 \Rightarrow F(x) = E x^{\tau} = \frac{x}{2} + \frac{x}{2} F(x)^2$$

Solve for $F(x)$ & get the formula.

$F(x) =$

Proof of Theorem 9.10

① $E\tau = \infty$.

Pf: $F(x) = E x^{\tau} = \dfrac{1 - \sqrt{1-x^2}}{x}$

diff wrt x : $F'(x) = E \tau x^{\tau - 1}$

Put $x = 1$: $F'(1) = E\tau$

Find $F'(1)$ from formula & get $E\tau = +\infty$!

(2) Find $P(\tau = n)$

Know $E x^{\tau} = F(x) = \dfrac{1 - \sqrt{1 - x^2}}{x}$

$$E x^{\tau} = 1\, P(\tau=0) + x\, P(\tau=1) + x^2\, P(\tau=2) + \cdots$$

$$= \sum_{k=0}^{\infty} x^k \underbrace{P(\tau = k)}$$

Also $F(x) = E x^{\tau} = \dfrac{1 - \sqrt{1 - x^2}}{x}$. Taylor series & find a formula.