21-720 Measure Theory.

2020-10-09

- *This is an open book test. You may use your notes, homework solutions, books, and/or online resources (including software) while doing this exam.*
- *You may* not, *however, seek or receive assistance from a live human during the exam. This includes in person assistance, instant messaging, and/or posting on online forums / discussion boards. (Searching discussion boards is OK, though.)*
- *You must record yourself (audio, video and screen) and share it with me as instructed by email.*
- *Late submissions will not be accepted. Please ensure you allow yourself ample time to scan your exam, otherwise you will get zero credit.*
- *You have 90 minutes. The exam has a total of 4 questions and 40 points.*
- *Difficulty wise, Q1 $\approx$ Q2 $\approx$ Q3 $\lesssim$ Q4 Good luck ☺.*

Unless otherwise stated, we always assume the underlying measure space is (X, Σ, μ) and μ is a positive measure. The Lebesgue measure on $\mathbb{R}^d$ will be denoted by λ.

$\boxed{10}$ 1. Let A be the set of all real numbers between 0 and 1 which have the digit 7 appearing in their decimal expansion. Find $\lambda(A)$.

$\boxed{10}$ 2. Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be Lebesgue measurable. True or false: There exists a Borel measurable function $g\colon \mathbb{R}^d \to \mathbb{R}$ such that $f = g$ almost everywhere. Prove it, or find a counter example.

$\boxed{10}$ 3. Let X be a compact metric space, and μ be a finite measure on X. Let $f \in L^1(X)$. True or false: For every $\varepsilon > 0$ there exists $g\colon X \to \mathbb{R}$ which is continuous such that $\int_X |f - g| \, d\mu < \varepsilon$? Prove it, or find a counter example.

$\boxed{10}$ 4. Let $f\colon [0, \infty) \to \mathbb{R}$ be integrable with respect to the Lebesgue measure. True or false: There exists $x \in [0, 1)$ such that $\lim_{n\to\infty} f(x + n)$ exists? Prove it, or find a counter example.